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ANALYSIS IN BANACH SPACES

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Declaration

This doctoral thesis is submitted in partial fulfillment of the requirements for the degree of doctor (Ph.D.). The work submitted in this dissertation is the result of my own investigation, except where otherwise stated. I declare that I worked out this thesis independently and I quoted all used sources of information in accord with Methodical instructions about ethical principles for writing academic thesis. Moreover I declare that it has not already been accepted for any degree and is also not being concurrently submitted for any other degree.

In .... date .... Signature

Abstract

In the thesis, the field of Lipschitz mappings in Banach spaces is studied. Lipschitz mappings play an important role in contemporary nonlinear functional analysis. They are a reasonable relaxation of bounded linear mappings and therefore can be found in many theoretical as well as practical applications. A structure of a great interest in the field of Lipschitz mappings is the Lipschitz-free space over a metric space.

Lipschitz-free space is a Banach space which carries the complexity of the underlying metric space. Every Lipschitz mapping between two metric spaces induces a linear mapping between corresponding Lipschitz-free spaces. An introduction to the topic is presented within the chapter two of the thesis.

The structure of Lipschitz-free spaces over uniformly discrete spaces is studied in two of the attached articles. Most interest is given to the study of Schauder bases in Lipschitz-free spaces. In the first article there is shown how uniformly bounded, commuting Lipschitz retractions on the metric space are connected to the existence of Schauder basis on the Free space. A Schauder basis for the Free space of an integer lattice in any space with an unconditional basis is constructed, which applies particularly for the space c_0, where such a lattice represents a net. Also, it is shown that for fixed n, although nets in $\mathbb{R}^n$ do not need to be Lipschitz-equivalent, their Free spaces are always isomorphic.

In the third article, the topic of Schauder bases in Lipschitz-free spaces is further investigated. It introduces the classification extensional and retractional for some Schauder bases on the Free space. It is shown that even for a simple infinite uniformly discrete space the retractional basis does not need to exist. For the same space, an extensional Schauder basis was constructed. Lastly, a condition is set on a sequence of retractions, under which the resulting Schauder basis is conditional. With the use of basic topological facts it is proven that every retractional Schauder basis in Lipschitz-free space of a net in $\mathbb{R}^n$ is conditional.

There is also a contribution to the theory of Lipschitz mappings in nonseparable Banach spaces. In the second attached article there is proved that the nonseparable analogue of Gowers' theorem does not hold. Gowers proved that every real Lipschitz mappping from the sphere of c_0 stabilizes on the sphere of some infinite dimensional subspace. The presented result shows that if we consider a Lipschitz mapping from the sphere of $c_0(\Gamma)$, it does not need to stabilize on the sphere of some nonseparable subspace. Actually a counterexample is presented - a nonexpansive function, which varies more than $\frac{1}{4}$ on a sphere of any nonseparable subspace of $c_0(\Gamma)$.

Keywords: Lipschitz mapping, Lipschitz-free space, Banach space, Schauder basis.

Abstrakt

V práci je studováno téma Lipschitzovských funkcí v Banachových prostorech. Lipschitzovská zobrazení hrají důležitou roli v současné nelineární funkcionální analýze. Jsou rozumnou relaxací lineárních spojitých operátorů, a tak nacházejí celou škálu teoretických i praktických aplikací. Strukturou spadající do oblasti Lipschitzovských funkcí, která se dnes těší velkému vědeckému zájmu, jsou Lipschitzovsky volné prostory nad metrickým prostorem.

Lipschitzovsky volný prostor (zkráceně LV prostor) je Banachův prostor, který nese komplexitu metrického prostoru, nad kterým je definován. Každé lipschitzovské zobrazení mezi metrickými prostory indukuje lineární zobrazení mezi příslušnými LV prostory. V druhé kapitole práce je prezentován stručný úvod do teorie LV prostorů.

Ve dvou z přiložených článků je studována struktura LV prostorů nad stejnoměrně diskrétními prostory. Nejvíce pozornosti je věnováno studiu Schauderových bazí v LV prostorech. V prvním článku je ukázáno, jak souvisí stejnoměrně omezené, komutující Lipschitzovské retrakce na metrickém prostoru s existencí Schauderovy baze v LV prostoru. Je sestrojen konkrétní příklad Schauderovy baze na LV prostoru celočíselné mřížky v prostorech s bezpodmínečnou bazí; konkrétním příkladem může být mřížka v c_0, která je zde zároveň sítí. Pro libovolné pevné $n \in \mathbb{R}$ je ukázáno, že ačkoliv dvě sítě v $\mathbb{R}^n$ nemusí být Lipschitzovsky ekvivalentní, jejich příslušné LV prostory jsou lineárně isomorfní.

V třetím článku je téma Schauderových bazí na LV prostorech dále rozvinuto. Jsou definovány pojmy retrakční baze a rozšířená baze na LV prostorech. Je zde dokázáno, že dokonce pro poměrně jednoduchý stejnoměrně diskrétní prostor retrakční baze na příslušném LV prostoru nemusí existovat. Pro stejný prostor je však nalezena monotónní rozšířená baze. V poslední řadě je formulována podmínka na posloupnost retrakcí, za které je výsledná retrakční baze podmínečná. Využitím základních poznatků z topologie je dokázáno, že všechny retrakční baze na LV prostorech sítí v $\mathbb{R}^n$ jsou podmínečné.

V práci je také příspěvek k teorii Lipschitzovských funkcí v neseparabilních Banachových prostorech. V druhém přiloženém článku je dokázáno, že neseparabilní analogie Gowersovy věty neplatí. Gowers dokázal, že každá Lipschitzovská funkce z jednotkové sféry c_0 se stabilizuje na sféře nějakého nekonečně rozměrného podprostoru. Předložený výsledek ukazuje, že uvážíme-li Lipschitzovskou funkci ze sféry neseparabilního $c_0(\Gamma)$, nemusí se stabilizovat na sféře žádného neseparabilního podprostoru. Je zde předložen protipříklad: Kontrakce, jejíž hodnoty se na sféře libovolného neseparabilního podprostoru $c_0(\Gamma)$ různí o více než $\frac{1}{4}$.

Klíčová slova: Lipschitzovské zobrazení, Lipschitzovsky volný prostor, Banachův prostor, Schauderova baze.

Contents

Introduction

In functional analysis, nonlinear analysis plays a central role in contemporary research. While most of the linear theory seems to be well-understood, there are many open questions in nonlinear setting. For example, if we have a Banach space X with a property P and a Banach space Y which is isomorphic to X in some nonlinear sense - for instance Lipschitz, uniform or coarse - does Y have to posses the property P as well? We can pose even a more ambitious question: Do Y and X have to be linearly isomorphic? In our work we are going to concentrate on Lipschitz mappings and isomorphisms. In such a perspective one finds useful to define a so-called Lipschitz-free space over a metric space.

Lipschitz-free space is a Banach space canonically containing the underlying metric space. It reflects the metric structure of the underlying space, transferring it into linear setting. Lipschitz mappings between metric spaces naturally appear as linear maps between corresponding Free spaces. When transferring to linear mappings, the complexity of Lipschitz mappings cannot be dropped. It is carried over to the structure of Lipschitz-free spaces instead. That is also the reason why their structure is still so poorly understood. Lipschitz-free spaces have been used in solutions to several major problems to which definitions there was no need to introduce this intriguing structure. Their wide application pleads for an extensive study of the topic.

There are two chapters followed by three articles of the author in the thesis. The chapters play an introductory role to the results presented in the enclosed articles. Their aim is to recall some basic knowledge in the field and provide a little context to the presented articles. The chapters were meant to comprise neither all known results in the field nor bring new results and should therefore be read with this in mind.

The first chapter is more introductory and comprises more general results in nonlinear theory. Concretely, the concept of uniform continuity is defined and the famous theorem of Ribe is stated. Further, an introduction to the theory of Lipschitz mappings in Banach spaces and an introduction to the second enclosed article is provided.

In the second chapter, we introduce Lipschitz-free spaces, explore some of their basic properties and bring some of their practical as well as theoretical applications. We introduce the transport mass problem and we show the usage of Earth mover's distance as a dissimilarity measure between pictures in computer science. Some theoretical application is also mentioned - linearising Lipschitz mappings, existence of a linear lifting to a linear quotient in separable setting, stability of λ-bounded approximation property under Lipschitz homeomorphisms of Banach spaces.

The first and the third enclosed article are devoted to study some properties of Lipschitz-free spaces. Concretely we study Lipschitz-free spaces over some uniformly discrete spaces, often over nets in Banach spaces. One of the most important results is that Lipschitz-free spaces over nets in $C(K)$ spaces for K metrizable compactum have Schauder bases. Moreover, we show how to construct such a basis in the case of a grid net in c_0. Further, it is shown that there exists a uniformly discrete subset of $\mathbb{R}^2$ which Lipschitz-free space does not admit a retractional Schauder basis (Schauder basis arising from the Lipschitz retraction technique, used in the first article). However, this Free space admits a monotone Schauder basis. Last but not least, it is shown that a retractional Schauder basis on a Lipschitz-free space over a net in $\mathbb{R}^n$ has to be conditional.

The second article refers about existence of a real Lipschitz function from the sphere of $c_0(\Gamma)$, which does not stabilize on any nonseparable subspace of $c_0(\Gamma)$. It answers an open problem from a book by Guirao, Montesinos and Zizler [1].

Chapter 1

Uniform and Lipschitz mappings in Banach spaces

In nonlinear functional analysis one often studies linear (Banach) spaces and nonlinear mappings between or into them. In full generality, one cannot expect much from a continuous mapping between two Banach spaces - very little is preserved with no further assumptions. Indeed, it is known [2] that that two Banach spaces are homeomorphic if and only if their topological density is the same. Recall that the density of a topological space is the least cardinality of a dense subset. That means the class of homeomorphisms is not sufficient for studying Banach spaces. If we assume a little bit more about the mappings, we will see that already some properties are preserved. More concretely, uniform homeomorphism between two Banach spaces delivers a nice similarity between them:

Definition 1.1. Let (M,d), (N,ρ) be metric spaces and $f : M \to N$ a mapping. The mapping $\omega_f(t) : [0,\infty) \to [0,\infty)$ defined by

$$\omega_f(t) = \sup\left\{\rho\left(f(x),f(y)\right),\, x,y \in M,\, d(x,y) < t\right\}$$

is called modulus of continuity (of f). We say that f is uniformly continuous if $\lim_{t\to 0} \omega_f(t) = 0$. We say f is an uniform embedding, if it is injective, uniformly continuous and the map $f^{-1} : N \supseteq f(M) \to M$ is also uniformly continuous. If f is moreover onto N, we say it is an uniform homemorphism between M and N and these two spaces are uniformly homeomorphic.

Definition 1.2. A Banach space X is said to be crudely finitely representable in a Banach space Y if there exists $K > 0$, such that for every finite-dimensional subspace $E \subseteq X$ there exists a linear embedding $T : E \to Y$ with distortion less than K (i.e. $\|T\| \cdot \|T^{-1}\| < K$).

We can now state what we meant by that similarity between Banach spaces. It is captured in the theorem of Ribe from 1976.

Theorem 1.1 (Ribe, [3])**.** *Let X,Y be Banach spaces. If X and Y are uniformly homeomorphic, then X is crudely finitely representable in Y and Y is crudely finitely representable in X.*

It means that local properties of Banach spaces (properties which involve only finitely many vectors in their definition) are preserved by uniform homeomorphism. Another immediate corollary of the theorem is that ℓ_p is not uniformly homeomorphic to ℓ_q or c_0 if $p \neq q$.

In this direction, there are even stronger results: We have that ℓ_p spaces wíth $1 < p < \infty$ have a unique uniform structure, i.e. if a Banach space is uniformly homeomorphic to ℓ_p, $1 < p < \infty$, then it is linearly isomorphic to it. A comprehensive list of Banach spaces with unique uniform structure is presented in [4]. Despite being useful, uniform homeomorphisms are not sufficient to describe completely Banach spaces, not even the class of separable ones (see [5], [6]). Let us therefore focus on a stronger class of mappings, namely the class of Lipschitz mappings.

Definition 1.3. Let (M,d), (N,ρ) be metric spaces. A mapping $f : M \to N$ is called Lipschitz whenever there exists $K > 0$ such that for all $x,y \in M$ holds

$$\rho\left(f(x),f(y)\right) \leq Kd(x,y).$$

Smallest $K > 0$ satisfying the above inequality is called Lipschitz constant of f and denoted $\operatorname{Lip} f$. If f is injective and f^{-1} is also Lipschitz, we say f is a bi-Lipschitz mapping or a Lipschitz embedding from M to N. If f is moreover onto N, we say f is a Lipschitz isomorphism between M and N and that the two spaces are Lipschitz isomorphic or Lipschitz equivalent. We say such isomophism f has distortion at most $K \geq 1$ if $\operatorname{Lip} f \cdot \operatorname{Lip} f^{-1} \leq K$.

Lipschitz isomorphisms carry more information about the structure of the spaces than uniform homeomorphisms. Straight examples can be the properties reflexivity, RNP or Asplund, which are stable under Lipschitz isomorphisms but not under uniform one [4]. Actually a lot of linear properties is preserved by Lipschitz isomorphisms of Banach spaces. For some of them there were even purely metric characterisations found. One can therefore define such properties in metric spaces. Such examples are e.g. superreflexivity [7], Rademacher type [8] and cotype [9], RNP [10] or reflexivity [11].

Important question concerning Lipschitz mappings is whether two Banach spaces which are Lipschitz isomorphic to each other are linearly isomorphic. Aharoni and Lindenstrauss [12] proved this does not need to be the case if the spaces are nonseparable. They presented a Banach space which is Lipschitz-equivalent to $c_0(\Gamma)$ with Γ of continuum cardinality which is not linearly isomorphic to $c_0(\Gamma)$. Since then many similar examples were derived from this original one. All of these examples share similar properties: the spaces are nonseparable and their Lipschitz equivalence to some other spaces is based on existence of a Lipschitz lifting to a certain linear quotient. In separable setting, the problem is still open and moreover, due to [13], the same technique using Lipschitz lifting cannot work. More specifically, Godefroy and Kalton showed that if a linear quotient map to a separable Banach space X has a Lipschitz right inverse, then it has a linear right inverse.

In one of the attached articles, we are studying the problem of distorting Lipschitz mappings on the sphere of a Banach space. In this article, the term distortion is used in a different meaning than in definition 1.3. We say a function $f : S_X \to \mathbb{R}$ from a sphere of an infinite-dimensional Banach space X is a distortion whenever there exists $\varepsilon > 0$ such that for any infinite-dimensional subspace $Y \subseteq X$ there exist points $x,y \in S_Y$, such that $|f(x) - f(y)| > \varepsilon$. In this direction, James proved that there is no equivalent norm on X which can be a distortion for $X = c_0$ or $X = \ell_1$. For reflexive ℓ_p spaces it is not true, as Odell and Schlumprecht show in [14]. If we look at the results for Lipschitz functions, we need to mention a famous result by Gowers [15], who showed that every real Lipschitz function from the unit sphere of c_0 stabilizes on a sphere of some infinite dimensional subspace. Precisely, for every $\varepsilon > 0$ and every Lipschitz function $f : S_{c_0} \to \mathbb{R}$, there exists an infinite dimensional subspace $Y \subseteq c_0$, such that for any $x,y \in S_Y$ we have $|f(x) - f(y)| \leq \varepsilon$. Clearly, the same would hold for any space containing c_0. When generalizing the Lipschitz distortion problem to nonseparable setting, it comes natural to allow the "stabilizing subspace" to be only of the same metric density as the original space. We studied this question on the space $c_0(\Gamma)$ for any uncountable cardinal Γ and we have constructed an example of a Lipschitz function on the sphere of $c_0(\Gamma)$, which is a distortion on any nonseparable subspace of $c_0(\Gamma)$. Formally, for any uncountable cardinal Γ, we constructed a symmetric 1-Lipschitz function $F : S_{c_0(\Gamma)} \to \mathbb{R}$, such that for any nonseparable subspace $Y \subseteq c_0(\Gamma)$ and any $x,y \in S_Y$ we have $|F(x) - F(y)| \geq \frac{1}{4}$.

Chapter 2

Lipschitz-Free Spaces

In this chapter we will define Lipschitz-free spaces and show some of their applications in mathematics and computer science. Let (M,d) be a metric space with a distinguished point (denoted $0 \in M$ for convenience). We call the triple $(M,d,0)$ a pointed metric space. On the set

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R},\ f(0) = 0,\ f \text{ Lipschitz}\}$$

we define a real-valued mapping

$$\|f\| = \sup_{x,y \in M,\ x \neq y} = \frac{|f(x) - f(y)|}{d(x,y)}.$$

It is not difficult to check $\|\cdot\|$ is a norm and that $(\mathrm{Lip}_0(M),\|\cdot\|)$ is a Banach space.

Definition 2.1. Let (M,d) be a metric space. A molecule on M is a function $m : M \to \mathbb{R}$ with finite support and such that $\sum_{x \in M} m(x) = 0$. For distinct points $p,q \in M$, we set $m_{pq} = \chi_{\{p\}} - \chi_{\{q\}}$ and on the space of molecules on M we define the seminorm

$$\|m\| = \inf \left\{ \sum_{i=1}^{n} |a_i| d(p_i,q_i) :\ m = \sum_{i=1}^{n} a_i m_{p_i,q_i} \right\}.$$

The completion of the space of all molecules on M modulo null vectors we call the Lipschitz-free space over M and denote it $\mathcal{F}(M)$.

We have this important dual characterization.

Theorem 2.1 ([16])**.** *Let $(M,d,0)$ be a pointed metric space. Then $\mathcal{F}(M)^*$ is linearly isometric to* $\mathrm{Lip}_0(M)$.

Proof. Define $T : \mathcal{F}(M)^* \to \mathrm{Lip}_0(M)$ as $Tx'(p) = x'(m_{p,0})$, $x' \in \mathcal{F}(M)^*$, $p \in M$. Clearly T is linear and for every $p,q \in M$ we have

$$|Tx'(p) - Tx'(q)| = |x'(m_{p,0}) - x'(m_{q,0})| = |x'(m_{p,q})| \leq \|x'\| d(p,q),$$

as $\|m_{p,q}\| \leq d(p,q)$. So T is a non-expansive linear operator. Define now $P : \mathrm{Lip}_0(M) \to \mathcal{F}(M)^*$ as $Pf(m) = \sum_{p \in M} m(p) f(p)$ for $f \in \mathrm{Lip}_0(M)$ and $m \in$

$\mathcal{F}(M)$ a molecule. Pick an arbitrary molecule $m \in \mathcal{F}(M)$ and suppose it can be expressed as $m = \sum_{i=1}^{n} a_i m_{p_i,q_i}$ for some eligible $a_i \in \mathbb{R}$,p_i,$q_i \in M$,$n \in \mathbb{N}$. Then

$$|Pf(m)| = \left|\sum_{p\in M} m(p)f(p)\right| = \left|\sum_{i=1}^{n} a_i \left(f(p_i) - f(q_i)\right)\right| \leq \|f\| \sum_{i=1}^{n} |a_i| d(p_i,q_i).$$

If we take infimum over all possible expressions of m, we get that $|Pf(m)| \leq \|f\|\|m\|$. Hence we can extend Pf to entire $\mathcal{F}(M)$ and we get that $Pf \in \mathcal{F}(M)^*$ for any $f \in \mathrm{Lip}_0(M)$ and that P is nonexpansive. A routine computation shows that $PT = TP = \mathrm{Id}$ and the proof is finished. □

If M is a pointed metric space and $0 \in S \subseteq M$ a subset containing the origin, we see that the restriction operator $R : \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(S)$ is a linear quotient. One could ask if $\mathrm{Lip}_0(S)$ is not actually a (complemented) subspace of $\mathrm{Lip}_0(M)$. In general we can say only that $\mathrm{Lip}_0(S)$ is isometric to a subset of $\mathrm{Lip}_0(M)$:

Lemma 2.2. *[17] Suppose (M,d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$. Then the following formula defines a K-Lipschitz function $\hat{g} : M \to \mathbb{R}$ such that $\hat{g}|_S = g$.*

$$\hat{g}(x) = \inf_{y\in S} \{g(y) + Kd(x,y)\}. \tag{2.1}$$

It is clear that the mapping $g \to \hat{g}$ is in general not linear in g. Later in this section we will see that $\mathcal{F}(S)$ is actually a linear subspace of $\mathcal{F}(M)$ and obviously if it is complemented, then $\mathrm{Lip}_0(S)$ can be seen as a linear (complemented) subspace of $\mathrm{Lip}_0(M)$.

In the definition of a Lipschitz-free space we needed to factor out null vectors to obtain a normed linear space. From Theorem 2.1 follows that $\|\cdot\|$ is actually a norm, wherefore there was nothing to factor out. Indeed, using the extension formula 2.1 we can find for any nonzero molecule m a 1-Lipschitz function $f \in \mathrm{Lip}_0(M)$ such that $0 < \sum_{p\in M} m(p)f(p) \leq \|m\|$.

Another immediate consequence of Theorem 2.1 and the fact that $\mathcal{F}(M)$ was defined regardless of the choice of 0 is that changing the base point in the space M does not change the Banach space structure of $\mathrm{Lip}_0(M)$.

Finally, it follows that for any two points $x,y \in M$ holds $\|m_{x,y}\| = d(x,y)$, since the function $d(\cdot,0)$ belongs to $\mathrm{Lip}_0(M)$ and is of norm one.

In many articles including those attached here, one uses an "external" definition of Lipschitz-free space: For a pointed metric space $(M,d,0)$ and δ_x denoting the Dirac evaluation functional at $x \in M$, Lipschitz-free space is the closure of span $\{\delta_x,\, x \in M\} \subseteq \mathrm{Lip}_0^*(M)$ in the norm of $\mathrm{Lip}_0^*(M)$. It is not difficult to prove that the two definitions coincide. First, any molecule m can be identified with the linear combination $\sum_{p\in M} m(p)\delta_p$ and any linear combination of Diracs $\sum_{i=1}^{n} a_i\delta_{x_i}$, $x_i \neq 0$ can be viewed as a molecule

$$m(z) = \begin{cases} a_i & z = x_i\,, \\ -\sum_{i=1}^{n} a_i & z = 0\,, \\ 0 & \text{else}\,. \end{cases}$$

Using Theorem 2.1 we get that the norms on both spaces coincide, also, both definitions of Lipschitz-free space agree. We include two examples of finite-dimensional Lipschitz-free spaces, which are used in computer science.

Example 2.2 (Transport mass problem). Suppose we have factories with locations $x_1,...,x_n$ producing $v_1,...,v_n$ units of some product and stores with locations $y_1,...,y_m$ where we want to deliver $w_1,...,w_m$ units of that product. Suppose $\sum_{i=1}^n v_i = \sum_{j=1}^m w_i$ and that distance metric function d on $M = \{x_1,...,x_n\} \cup \{y_1,...,y_m\}$ is given. The task is to minimize the total cost of transport of goods from factories to stores given by $\sum_{i=1}^n \sum_{j=1}^m h(i,j)d(x_i,y_j)$, where $h(i,j) \geq 0$ denotes the amount of units transported from x_i to y_j, under the conditions $\sum_{i=1}^n h(i,j) = w_j$ and $\sum_{j=1}^m h(i,j) = v_i$ for all possible i,j (meaning all units have to be transported from factories to stores in a way that every store's demand is met). Then the minimal transport cost equals $\|m\|$, where m is a molecule defined on (M,d) by

$$m(z) = \begin{cases} v_i & z = x_i,\ i \in \{1,...,n\}\ , \\ -w_j & z = y_j,\ j \in \{1,...,m\}\ , \end{cases}$$

and the values of the optimal transport plan $h : \{x_1,...,x_n\} \times \{y_1,...,y_m\} \to \mathbb{R}$ are exactly the constants a_i from the definition 2.1, where the infimum is attained.

Clearly, the Lipschitz-free space $\mathcal{F}(M)$ in this example is representing the space of all possible assignments of amounts $\nu_1,...,\nu_{n+m} \in \mathbb{R}$ over M (hence the numbers and positions of factories and stores within points of M), such that the amount of manufactured goods is equal to the one stored.

The important practical question is, how to find the minimal transport cost, resp. some minimal transport plan. In some special cases, for example when all points of M lie on the same line, the solution is easy to find and its asymptotical complexity is low. Concretely we have:

Proposition 2.3. *Let $M = \{x_1,x_2,...,x_n\}$ be an n-point subset of the real line. Then $\mathcal{F}(M)$ is linearly isometric to ℓ_1^{n-1} and for every $m \in \mathcal{F}(M)$, $\|m\|$ can be computed in linear time (with respect to n).*

Proof. Without loss of generality we can assume that $x_1 < x_2 < ... < x_n$. Let us define a mapping $T : \ell_1^{n-1} \to \mathcal{F}(M)$ as $Te_i = \frac{1}{d(x_i,x_{i+1})} m_{x_{i+1},x_i}$. Then for every $x = \sum_{i=1}^{n-1} a_i e_i \in \ell_1^{n-1}$ we have

$$\|Tx\| = \left\| \sum_{i=1}^{n-1} \frac{a_i}{d(x_i,x_{i+1})} m_{x_{i+1},x_i} \right\| \leq \sum_{i=1}^{n-1} |a_i| = \|x\|.$$

On the other hand, every element $m \in \mathcal{F}(M)$ can be uniquely represented as a sum $m = \sum_{i=1}^{n-1} a_i m_{x_{i+1},x_i}$, which yields that the mapping $T^{-1} : \mathcal{F}(M) \to \ell_1^{n-1}$, $T^{-1}m = \sum_{i=1}^{n-1} a_i d(x_i,x_{i+1}) e_i$ is well-defined. Clearly $TT^{-1} = T^{-1}T = \mathrm{Id}$. Fix now $m \in \mathcal{F}(M)$ and define a function $f = f_m : M \to \mathbb{R}$ inductively by $f(x_1) = 0$ and $f(x_{i+1}) - f(x_i) = d(x_{i+1},x_i) \operatorname{sgn} a_i$, $i \in \{1,...,n-1\}$. Then

$$\begin{aligned} \|T^{-1}m\| &= \left\| \sum_{i=1}^{n-1} a_i d(x_i,x_{i+1}) e_i \right\| = \sum_{i=1}^{n-1} |a_i d(x_i,x_{i+1})| \\ &= -a_1 f(x_1) + \sum_{i=1}^{n-2} (a_i - a_{i+1}) f(x_{i+1}) + a_{n-1} f(x_n) \\ &= \sum_{i=1}^{n} m(x_i) f(x_i) \overset{2.1}{\leq} \|m\|, \end{aligned}$$

since $f \in \mathrm{Lip}_0(M,d,x_1)$ and $\|f\| = 1$. We get that T is a linear isometry.

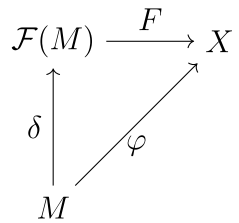
From the previous follows that one can compute the norm of $m \in \mathcal{F}(M)$ in linear time. Indeed, we have $\|m\| = \sum_{i=1}^{n-1} |a_i| d(x_{i+1},x_i)$, where the constants a_i are the uniquely given constants satisfying $m = \sum_{i=1}^{n-1} a_i m_{x_{i+1},x_i}$. But one can find these constants easily in linear time as $a_1 = -m(x_1)$, $a_i = a_{i-1} - m(x_i)$, $i \in \{2,...,n-1\}$. □

In general, Godard [18] gives a nice characterization of metric spaces, which Lipschitz-free spaces are isomorphic to a subspace of L_1. We have the following:

Example 2.3 ([18]). The Lipschitz-free space $\mathcal{F}(\mathbb{R})$ is linearly isometric to L_1. For any discrete infinite set $M \subseteq \mathbb{R}$, the Lipschitz-free space $\mathcal{F}(M)$ is linarly isometric to ℓ_1. A graph tree T with weighted edges endowed with the shortest path metric is isometric to ℓ_1^n, where $n = \mathrm{card}(T) - 1$.

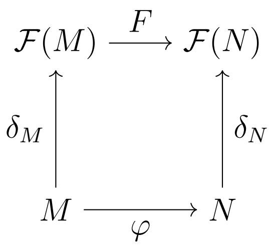
To assume that points of M lie on a line or that their distances correspond to a tree metric is often not possible. However, one can still get some estimates on the minimal transport cost in low time complexity. The following theorems show more about the structure of Lipschitz-free spaces, from which the estimates follow. We state them in the terms of the "external" definition of Lipschitz-free spaces. Recall that we denote δ the isometry which sends every point of M to the Dirac evaluation functional $\delta_x \in \mathcal{F}(M) \subseteq \mathrm{Lip}_0(M)^*$ at x (or equivalently, to the molecule $m_{x,0}$). The following theorem is known as the universal property of Lipschitz free spaces.

Theorem 2.4. *Let $(M,d,0)$ be a pointed metric space, X a Banach space and $\varphi : M \to X$ a Lipschitz mapping satisfying $\varphi(0) = 0$. Then there is a unique linear mapping $F : \mathcal{F}(M) \to X$ with $\|F\| = \|\varphi\|_{\mathrm{Lip}}$ such that the following diagram commutes.*



Proof. Define F on $\mathrm{span}\{\delta_x,\, x \in M\}$ as a linear extension of φ. For every $a \in \mathrm{span}\{\delta_x,\, x \in M\}$, there is a linear functional $f \in B_{X^*}$ such that $\|F(a)\|_X = f(F(a))$. But $f \circ \varphi \in \mathrm{Lip}_0(M)$, which implies $\|f \circ \varphi\|_{\mathrm{Lip}} \le \|\varphi\|_{\mathrm{Lip}}$ and therefore $\|F(a)\| \le \|a\| \cdot \|\varphi\|_{\mathrm{Lip}}$. So F is bounded on $\mathrm{span}\{\delta_x,\, x \in M\}$ and so we can extend it to the closure $\overline{\mathrm{span}}\{\delta_x,\, x \in M\} = \mathcal{F}(M)$. □

Corollary 2.5. *Let M,N be pointed metric spaces and $\varphi : M \to N$ a Lipschitz mapping with $\varphi(0) = 0$. Then there exists a unique linear mapping $F : \mathcal{F}(M) \to \mathcal{F}(N)$ with $\|F\| = \mathrm{Lip}\,\varphi$ such that the following diagram commutes.*



Proof. It suffices to apply the previous lemma to the mapping $\delta_N \circ \varphi : M \to \mathcal{F}(N)$. $\square$

If $N \subseteq M$, we can view the space $\mathcal{F}(N)$ as a linear subspace of $\mathcal{F}(M)$. If is moreover N a Lipschitz retract of M, then $\mathcal{F}(N)$ is complemented in $\mathcal{F}(M)$. Analogously, if M,N are Lipschitz isomorphic with distortion at most K, then $\mathcal{F}(M)$ and $\mathcal{F}(N)$ are linearly isomorphic with distortion at most K.

The last assertion indeed gives an estimate for the minimal transport cost of some transport mass problem. Namely, if (M,d) and (N,ρ) are metric spaces and $\varphi : M \to N$ is a Lipschitz isomorphism with distortion at most $K \geq 1$, then for any transport mass problem assignment $m \in \mathcal{F}(M)$ one knows that the minimal transport cost of m, i.e. $\|m\|$, is at least $1/K$-times the cost of Fm and at most K-times the cost of Fm, where F is the linear isomorphism from Corollary 2.5 corresponding to the mapping φ.

Lipschitz-free spaces are often used for linearizing Lipschitz maps between metric spaces. This is particularly interesting in the case when the underlying metric spaces are Banach spaces. Firstly, it means that whenever two Banach spaces X,Y are Lipschitz isomorphic, then their corresponding Free spaces are linearly isomorphic. This can be used in answering to the following questions:

1. Suppose X,Y are Banach spaces which are Lipschitz isomorphic to each other. Suppose X has some property P. Does it follow that Y has also property P?

2. Suppose X,Y are Banach spaces which are Lipschitz isomorphic to each other. Are X and Y linearly isomorphic?

Whenever we have that a Banach space X has property P if and only if its Free space $\mathcal{F}(X)$ has P we get immediately that property P is stable under Lipschitz isomorphism. An example of such a property is λ-bounded approximation property [13]:

Definition 2.4. Let $\lambda \geq 1$. We say that a Banach space X has λ-bounded approximation property, if for every $\epsilon > 0$ and every compact set $K \subseteq X$, there exists a finite rank linear operator $T : X \to X$ such that for each $x \in K$ we have $\|Tx - x\| < \epsilon$ and $\|T\| \leq \lambda$.

There is a well-known open question whether every Banach space that is Lipschitz isomorphic to ℓ_1 is linearly isomorphic to it. To this side it is known that if the Free space $\mathcal{F}(\ell_1)$ is complemented in its bidual, then the answer to the previous question is yes.

The next example is a simplified version of a process of measuring dissimilarity between images. Usually, the image is compressed for the sake of efficiency of used methods. The easiest way to do it is to quarter the image into bins and then describe the image as a sequence of brightness values (grey-scaled images) or as a sequence of RGB valued vectors.

Example 2.5 (Image retrieval). Suppose we have a set of images, each represented as a sequence of brightness intensity values on a bin-grid $M = \{x_1,x_2,...,x_n\}$ for some given $n \in \mathbb{N}$. Let d_{ij} be the distance between bins x_i and x_j. For two images S,T with brightness values $s_1,...,s_n$ and $t_1,...,t_n$, one defines the so-called

Earth mover's distance between them in the following way: First, we need to find values $f_{ij} \geq 0$, such that the following value is minimized

$$\mathrm{WORK}(S,T,f) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} d_{ij},$$

while the following constraints are satisfied:

$$\sum_{i=1}^{n} f_{ij} \leq t_j,\ j \in \{1,...,n\}\,, \tag{2.2}$$

$$\sum_{j=1}^{n} f_{ij} \leq s_i,\ i \in \{1,...,n\}\,, \tag{2.3}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} = \min\left\{\sum_{i=1}^{n} s_i, \sum_{j=1}^{n} t_j\right\}. \tag{2.4}$$

Once $\{f_{ij}\}_{i,j=1}^{n}$ is found, then the Earth mover's distance between S and T is defined as

$$\mathrm{EMD}(S,T) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} d_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}}.$$

For a specific example, when each picture has total brightness intensity equal to 1 and d is a metric on M, then EMD becomes a metric [19]. Namely EMD is a metric on $X = \{S : \sum_{i=1}^{n} s_i = 1\}$. In this example, Lipschitz-free space over M would consist of all pairs of pictures (S,T) such that the total brightness intensity of S is the same as of T modulo all pairs in which the two pictures coincide. Here we assume the linear structure on $\{(S,T) : \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i\}$ given by considering every picture S as a brightness function on M, i.e. $S : M \to \mathbb{R}$, $S(x_i) = s_i$. Hence $\alpha(S_1,T_1) + \beta(S_2,T_2) = (\alpha S_1 + \beta S_2, \alpha T_1 + \beta T_2)$. The norm is given here by $\|(S,T)\|_W = \min_f \mathrm{WORK}(S,T,f)$, where f satisfies the constraints 2.2 - 2.4. Analogously, the set $\mathcal{F}(M)$ can be thought of as all pairs of pictures (S,T), such that the total brightness intensity is the same among S and T, modulo all $\|\cdot\|_W$-null vectors.

In the aforementioned example, one often needs to compute the EMD measure between two pictures. There are approximative methods [20] to compute EMD in linear time using embeddings into L_1. However, in [21] was shown for the grid $M = \{0,1,...,n\}^2$ in $\mathbb{R}^2$ that embedding of $\mathcal{F}(M)$ into L_1 incurs distortion $\Omega(\sqrt{\log n})$.

The last result has also theoretical implications. It follows that $\mathcal{F}(\mathbb{R}^2)$ does not embed into L_1. Since by [22] for every set $M \subseteq \mathbb{R}^n$ with nonempty interior the space $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(\mathbb{R}^n)$, we have also that $\mathcal{F}(M)$ does not embed into L_1 for any such set when $n \geq 2$.

It is still an open question whether $\mathcal{F}(\mathbb{R}^2)$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^3)$.

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List of publications

The following publications are attached in the chronological order of their appearance:

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2. Hájek, Petr; Novotný, Matěj. Distortion of Lipschitz functions on $c_0(\Gamma)$. Proc. Amer. Math. Soc. 146 (2018), no. 5, 2173–2180. https://doi.org/10.1090/proc/13945
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Some remarks on the structure of Lipschitz-free spaces*

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Abstract

We give several structural results concerning the Lipschitz-free spaces $\mathcal{F}(M)$, where M is a metric space. We show that $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$, where $\Gamma = \text{dens}(M)$. If $\mathcal{N}$ is a net in a finite dimensional Banach space X, we show that $\mathcal{F}(\mathcal{N})$ is isomorphic to its square. If X contains a complemented copy of ℓ_p, c_0 then $\mathcal{F}(\mathcal{N})$ is isomorphic to its ℓ_1-sum. Finally, we prove that for all $X \cong C(K)$ spaces, where K is a metrizable compact, $\mathcal{F}(\mathcal{N})$ are mutually isomorphic spaces with a Schauder basis.

1 Introduction

Let (M, d) be a metric space and $0 \in M$ be a distinguished point. The triple $(M, d, 0)$ is called pointed metric space. By $\text{Lip}_0(M)$ we denote the Banach space of all Lipschitz real valued functions $f : M \to \mathbb{R}$, such that $f(0) = 0$. The norm of $f \in \text{Lip}_0(M)$ is defined as the smallest Lipschitz constant $L = \text{Lip}(f)$ of f, i.e.

$$\text{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)},\ x, y \in M,\ x \neq y\right\}.$$

The Dirac map $\delta : M \to \text{Lip}_0(M)^*$ defined by $\langle f, \delta(p)\rangle = f(p)$ for $f \in \text{Lip}_0(M)$ and $p \in M$ is an isometric embedding from M into $\text{Lip}_0(M)^*$.

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Note that $\delta(0) = 0$. The closed linear span of $\{\delta(p),\ p \in M\}$ is denoted $\mathcal{F}(M)$ and called the Lipschitz-free space over M (or just free space, for short). Clearly,

$$\|m\|_{\mathcal{F}(M)} = \sup\left\{\langle m, f\rangle : f \in \mathrm{Lip}_0(M), \|f\| \leq 1\right\}$$

It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as the canonical predual of $\mathrm{Lip}_0(M)$, i.e. $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$ holds isometrically ([33] Chapter 2 for details).

Lipschitz free spaces have gained importance in the non-linear structural theory of Banach spaces after the appearance of the seminal paper [13] of Godefroy and Kalton, and the subsequent work of these and many other authors e.g. [19], [20], [21], [22], [14], [24], [16], [17], [31], [23], [5], [4], [10], [6], [29] [7], [8], [9]. Free spaces can be used efficiently for constructions of various examples of Lipschitz-isomorphic Banach spaces X, Y which are not linearly isomorphic. To this end, structural properties of their free spaces $\mathcal{F}(X)$, as well as free spaces of their subsets, enter the game. For example, in the separable setting, $\mathcal{F}(X)$ contains a complemented copy of X [13], and it is isomorphic to its ℓ_1-sum. On the other hand, if $\mathcal{N}$ is a net in X then $\mathcal{F}(\mathcal{N})$ is a Schur space [20] and it has the approximation property.

A comprehensive background on free spaces of metric spaces can be found in the book of Weaver [33]. There are several surveys exposing the applications of this notion to the nonlinear structural theory of Banach spaces, in particular [19], [15].

Our first observation in this note is that $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$, where Γ is the density character of an arbitrary infinite metric space M. Our proof could be adjusted also to the case $\Gamma = \omega_0$, which is one of the main results in [5].

The main purpose of this note is to prove several structural results, focusing mainly on the case when M is a uniformly discrete metric space, in particular a net $\mathcal{N}$ in a Banach space X. Our results run parallel (as we have realized during the preparation of this note) to those of Kaufmann [23], resp. Dutrieux and Ferenczi [10] which are concerned with the bigger (in a sense) space $\mathcal{F}(X)$. However, the space $\mathcal{F}(\mathcal{N})$ is only the linear quotient of $\mathcal{F}(X)$, so the results are certainly not formally transferable. In particular, the discrete setting prohibits the use of the "scaling towards zero" arguments (used e.g. in [23]), which leads to complications in proving that our free spaces are linearly isomorphic to their squares, or even ℓ_1-sums. We are able to show these facts at least for nets in finite dimensional Banach spaces and all classical Banach spaces. Surprisingly, the proofs for the finite dimensional case and the infinite dimensional case are rather different.

Our main technical result is that $\mathcal{F}(\mathcal{N})$ has a Schauder basis for all nets in $C(K)$ spaces, K metrizable compact. The constructive proof is obtained in c_0, and the result is then transferred into the $C(K)$ situation by using the abstract theory developed in the first part of our note.

Let us start with some definitions and preliminary results. Let $N \subset M$ be metric spaces, and assume that the distinguished point $0 \in M$ serves as a distinguished point in N as well. Then the identity mapping leads to the canonical isometric embedding $\mathcal{F}(N) \hookrightarrow \mathcal{F}(M)$ ([33] p.42). In order to study the complementability properties of this subspace, one can rely on the theory of quotients of metric space, as outlined in [33] p.11 or [23]. For our purposes we will outline an alternative (but equivalent) description of the situation.

Definition 1. *Let $N \subset M$ be metric spaces, $0 \in N$. We denote by*

$$Lip_N(M) = \{f \in Lip_0(M) : f|_N = 0\}.$$

It is clear that $Lip_N(M)$ is a closed linear subspace of $Lip_0(M)$, which is moreover w^*-closed. Indeed, by the general perpendicularity principles ([11] p.56) we obtain

$$Lip_N(M) = \mathcal{F}(N)^{\perp}, \;\; \mathcal{F}(N) = Lip_N(M)_{\perp}$$

Hence there is a canonical isometric isomorphism

$$Lip_N(M) \cong (\mathcal{F}(M)/\mathcal{F}(N))^*$$

Since the space of all finite linear combinations of Dirac functionals is linearly dense in $\mathcal{F}(M)$, resp. also in $\mathcal{F}(N)$, it is clear that the image of finite linear combinations of Dirac functionals supported outside the set N, under the quotient mapping $\mathcal{F}(M) \to \mathcal{F}(M)/\mathcal{F}(N)$ is linearly dense. Moreover, it is nonzero for nontrivial combinations.

Definition 2. *If $\mu = \sum_{j=1}^{n} a_j\delta_{t_j} : t_j \in M \setminus N$ then we let*

$$\|\mu\|_{\mathcal{F}_N(M)} = \sup\langle \mu, f\rangle, \;\; f \in Lip_N(M), \; \|f\| \leq 1.$$

$$\mathcal{F}_N(M) = \overline{\left\{\mu = \sum_{j=1}^{n} a_j\delta_{t_j} : t_j \in M \setminus N\right\}}^{\|\cdot\|_{\mathcal{F}_N(M)}}.$$

i.e. we complete the space of finite sums of Dirac functionals with respect to the duality

$$\langle \mathcal{F}_N(M), Lip_N(M)\rangle.$$

Clearly, our definition gives an isometric isomorphism

$$\mathcal{F}_N(M) \cong \mathcal{F}(M)/\mathcal{F}(N)$$

Proposition 1. *Let $N \subset M$ be metric spaces. If there exists a Lipschitz retraction $r : M \to N$ then*

$$\mathcal{F}(M) \cong \mathcal{F}(N) \oplus \mathcal{F}_N(M).$$

This follows readily from the alternative description using metric quotients (e.g. in [23] Lemma 2.2) using the fact that $\mathcal{F}_N(M) \cong \mathcal{F}(M/N)$.

We say that the metric space (M,d) is δ-uniformly discrete if there exists $\delta > 0$ such that $d(x,y) \geq \delta, x, y \in M$. The metric space is uniformly discrete if it is δ-uniformly discrete for some $\delta > 0$.

If $\alpha, \beta > 0$ we say that a subset $N \subset M$ is a (α,β)-net in M provided it is α-uniformly discrete and $d(x,N) < \beta, x \in M$.

It is easy to see that every maximal δ-separated subset $N \subset M$, which exists due to the Zorn maximal principle, is automatically a $(\delta, \delta+\varepsilon)$-net, for any $\varepsilon > 0$.

Proposition 2. *Let $(M,d,0)$ be a pointed metric space, $K > 0$, $\{M_\alpha\}_{\alpha\in\Gamma}$ be a system of pairwise disjoint subsets of M, and $0 \in N \subset M \setminus \cup_{\alpha\in\Gamma} M_\alpha$. Suppose that for all $\beta \in \Gamma$ and all $x \in M_\beta$ holds*

$$d(x, \cup_{\alpha\in\Gamma,\alpha\neq\beta} M_\alpha) \geq K d(x,N).$$

Then

$$\mathcal{F}_N(N \cup \cup_{\alpha\in\Gamma} M_\alpha) \cong (\oplus_{\alpha\in\Gamma} \mathcal{F}_N(N \cup M_\alpha))_{\ell_1(\Gamma)}.$$

In particular, if $N = \{0\}$ then

$$\mathcal{F}(\{0\} \cup \cup_{\alpha\in\Gamma} M_\alpha) \cong (\oplus_{\alpha\in\Gamma} \mathcal{F}(\{0\} \cup M_\alpha))_{\ell_1(\Gamma)}.$$

Proof. The result is immediate as any collection of 1-Lipschitz functions $f_\alpha \in \mathrm{Lip}_N(N \cup M_\alpha)$ is the restriction of a $\frac{1}{K}$-Lipschitz function $f \in \mathrm{Lip}_N(N \cup \cup_{\alpha\in\Gamma} M_\alpha)$ ∎

Recall that the density character $\mathrm{dens}(M)$, or just density, of a metric space M is the smallest cardinal Γ such that there exists dense subset of M of cardinality Γ.

Let Γ be a cardinal (which is identified with the smallest ordinal of the same cardinality). By the cofinality $\mathrm{cof}(\Gamma)$ we denote the smallest ordinal α (in fact a cardinal) such that $\Gamma = \lim_{\beta<\alpha} \Gamma_\beta$, where Γ_β is an increasing ordinal sequence ([18] p.26).

2 Structural properties

Proposition 3. *Let M be a metric space of density $\mathrm{dens}(M) = \Gamma$. Then $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\Gamma)$.*

Proof. For convenience we may assume that $\Gamma > \omega_0$, because this case has been already proved in [5] (Our proof can be adjusted to this case as well). By ([32] Corollary 1.2) if $c_0(\Gamma) \hookrightarrow X^*$ then $\ell_1(\Gamma)$ is complemented in X. So it suffices to prove that $\mathrm{Lip}_0(M)$ contains a copy of $c_0(\Gamma)$. For every $n \in \mathbb{N}$ let M_n be some maximal $\frac{1}{2^n}$-separated set in M. Denote $\Gamma_n = |M_n|$. It is clear that $\mathrm{dens}(M) = \lim_{n\to\infty} \Gamma_n$, in the cardinal sense. In case when the cofinality $\mathrm{cof}(\Gamma) > \omega_0$, it is clear that $\Gamma_n = \Gamma$, for some $n \in \mathbb{N}$. In this case, let $\{f_\alpha : \alpha \in \Gamma_n\}$ be a transfinite sequence of 1-Lipschitz functions such that $f_\alpha(x_\alpha) = \frac{1}{2^{n+3}}$ and $\mathrm{supp}(f_\alpha) \subset B(x_\alpha, \frac{1}{2^{n+2}})$. Since the supports of f_α are pairwise disjoint it is clear that $\{f_\alpha\}_{\alpha\in\Gamma_n}$ is equivalent to the unit basis of $c_0(\Gamma)$ and the result follows. In the remaining case, we may assume that $\{\Gamma_{k_n}\}_{n=1}^{\infty}$ is a strictly increasing sequence of cardinals. Denote $M_n = \{x_\alpha^n\}_{\alpha\in\Gamma_{k_n}}$. Let $L_1 = M_1$. By induction we will construct sets $L_n \subset M_n$ as follows. Inductive step towards $n+1$. Consider the sets

$$A_{j,\alpha} = M_{n+1} \cap B(x_\alpha^j, \frac{1}{2^{k_j+1}}), \ \ j \le n, \alpha \in \Gamma_{k_j}$$

If there is some j, α so that $|A_{j,\alpha}| = \Gamma_{k_{n+1}}$ then we let $L_{n+1} = A_{j,\alpha}$. Otherwise we let

$$L_{n+1} = M_{n+1} \setminus \cup_{j\le n, \alpha\in\Gamma_{k_j}} A_{j,\alpha}$$

In either case we have $|L_{n+1}| = \Gamma_{k_{n+1}}$. By discarding suitable countable subsets of these sets L_n we can assume that

$$\operatorname{dist}(L_n, L_m) \ge \max\left\{\frac{1}{2^{k_n+1}}, \frac{1}{2^{k_m+1}}\right\}$$

To finish, let $\{f_\alpha^n : x_\alpha^n \in L_n, n \in \mathbb{N}\}$ be a transfinite sequence of 1-Lipschitz disjointly supported functions such that $f_\alpha^n(x_\alpha^n) = \frac{1}{2^{k_n+3}}$ and $\operatorname{supp}(f_\alpha^n) \subset B(x_\alpha^n, \frac{1}{2^{k_n+2}})$. This sequence is equivalent to the basis of $c_0(\Gamma)$, which finishes the proof. ∎

Theorem 4. *Let N, M be uniformly discrete infinite sets of the same cardinality such that $N \subset M$ is a net. Then $\mathcal{F}(N) \cong \mathcal{F}(M)$.*

Proof. Let $K > 0$ be such that $\max_{m\in M} \operatorname{dist}(m, N) \le K$ Let $r : M \to N$ be a retraction such that $d(x, r(x)) \le K$. As M is uniformly discrete, r is Lipschitz. By Proposition 1

$$\mathcal{F}(M) \cong \mathcal{F}(N) \oplus \mathcal{F}_N(M).$$

It is clear that $\mathcal{F}_N(M) \cong \ell_1(M \setminus N)$. By Proposition 3

$$\mathcal{F}(N) \cong \mathcal{F}(N) \oplus \ell_1(M) \cong \mathcal{F}(M).$$ ∎

Recall that all nets in a given infinite dimensional Banach space are Lipschitz equivalent ([26], or [1] p.239), hence their free spaces are linearly isomorphic. On the other hand, there are examples of non-equivalent nets in $\mathbb{R}^2$ ([28], [3] or [1] p.242), hence the next result is not immediately obvious.

Proposition 5. *Let $\mathcal{N}$, $\mathcal{M}$ be nets of the same cardinality $dens(M)$ in a metric space (M, d). Then $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{M})$.*

Proof. Suppose $\mathcal{N}$ is a (a,b)-net and $\mathcal{M}$ is a (c,d)-net in M, $a \le c$. Let $K = \mathcal{M} \cup \mathcal{N}$, and let $\mathcal{K} \subset K$ be maximal subset such that from each pair of points $x \in \mathcal{M}, y \in \mathcal{N}$ for which $d(x,y) < \frac{a}{4}$ we choose only one $x \in \mathcal{K}$. It is now clear that both $\mathcal{N}$ and $\mathcal{M}$ are bi-Lipschitz equivalent to a respective subset of $\mathcal{K}$. By Theorem 4, $\mathcal{F}(\mathcal{K}) \cong \mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$. ∎

Of course, the above proposition applies to any pair of nets in a given Banach space X, or its subset $S \subset X$ which contains arbitrarily large balls.

Lemma 6. *Let $Y = X \oplus \mathbb{R}$ be Banach spaces, $\mathcal{N}$ be a net in X and $\mathcal{M}$ be the extension of $\mathcal{N}$ into the natural net in Y. Denote $\mathcal{M}^+ = \mathcal{M} \cap X \oplus \mathbb{R}^+$, $\mathcal{M}^- = \mathcal{M} \cap X \oplus \mathbb{R}^-$.*

If $\mathcal{F}(\mathcal{N}) = \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$ and $\mathcal{F}(\mathcal{M}^+) = \mathcal{F}(\mathcal{M}^+) \oplus \mathcal{F}(\mathcal{M}^+)$ then $\mathcal{F}(\mathcal{M}) = \mathcal{F}(\mathcal{M}^+) = \mathcal{F}(\mathcal{M}) \oplus \mathcal{F}(\mathcal{M})$.

Proof. Thanks to Proposition 5 we are allowed to make additional assumptions on the form of the nets. Let us assume that $\mathcal{M} = \mathcal{N} \times \mathbb{Z}$, which immediately implies that $\mathcal{N} \cup \mathcal{M}^+$ is bi-Lipschitz equivalent with $\mathcal{M}^+$ (and $\mathcal{M}^-$) by translation. Denoting $P : Y \to X$ the canonical projection $P(x,t) = x$, we see that $P : \mathcal{M} \to \mathcal{N}$ is a Lipschitz retraction, so

$$\mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{N} \cup \mathcal{M}^+) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+)$$

and using Proposition 2

$$\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+) \oplus \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^-)$$

Since $\mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^+) \cong \mathcal{F}_{\mathcal{N}}(\mathcal{N} \cup \mathcal{M}^-)$ and $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$ the result follows.

■

Theorem 7. *Let $\mathcal{N}$ be a net in $\mathbb{R}^n$. Then $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$.*

Proof. For $n = 1$ it is well known [12] that $\mathcal{F}(\mathcal{N}) \cong \mathcal{F}(\mathcal{N}^+) \cong \ell_1 \cong \mathcal{F}(\mathcal{N}) \oplus \mathcal{F}(\mathcal{N})$.

Inductive step towards $n+1$. We may assume that $\mathcal{N} = \mathbb{Z}^{n+1}$ is the integer grid. Let us use the following notation (our convention is that $\mathbb{Z}^+ = \{1,2,3,\dots\}, \mathbb{Z}^- = \{-1,-2,\dots\}$).

$$\mathcal{L} = \mathbb{Z}^{n-1} \times \{0\} \times \{0\},\ \mathcal{L}_1 = \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \times \{0\},$$
$$\mathcal{L}_2 = \mathbb{Z}^{n-1} \times \{0\} \times \mathbb{Z}^+,\ \mathcal{L}_3 = \mathbb{Z}^{n-1} \times \mathbb{Z}^- \times \{0\}$$
$$\mathcal{M}^+ = \mathbb{Z}^{n-1} \times \mathbb{Z} \times \mathbb{Z}^+,\ \mathcal{M}_1 = \mathbb{Z}^{n-1} \times \mathbb{Z}^+ \times \mathbb{Z}^+,\ \mathcal{M}_2 = \mathbb{Z}^{n-1} \times \mathbb{Z}^- \times \mathbb{Z}^+$$

With this notation, we have the following bi-Lipschitz equivalence

$$\mathcal{L}_1 \cup \mathcal{L} \cup \mathcal{L}_2 \cong \mathcal{L}_1 \cup \mathcal{L} \cup \mathcal{L}_3.$$

By inductive assumption this implies

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2). \tag{1}$$

On the other hand, using Proposition 2 in various settings

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L}) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2),$$

$$\mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L} \cup \mathcal{L}_1}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1) \oplus \mathcal{F}_{\mathcal{L}}(\mathcal{L} \cup \mathcal{L}_2), \tag{2}$$

$$\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2\cup\mathcal{L}_3)\cong\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_3)\oplus\mathcal{F}_{\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_3}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2\cup\mathcal{L}_3)\cong \mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_3)\oplus\mathcal{F}_{\mathcal{L}}(\mathcal{L}\cup\mathcal{L}_2). \quad (3)$$

Hence, using the inductive assumption $\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_3)\cong\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1)$

$$\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2\cup\mathcal{L}_3)\cong\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1)\oplus\mathcal{F}_{\mathcal{L}}(\mathcal{L}\cup\mathcal{L}_2) \quad (4)$$

Comparing (2), (4) and using (1) we obtain

$$\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2\cup\mathcal{L}_3)\cong\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2)\cong\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2)\oplus\mathcal{F}(\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2) \quad (5)$$

By Lemma 6, in order to complete the inductive step, it suffices to prove that $\mathcal{F}(\mathcal{M}^+)=\mathcal{F}(\mathcal{M}^+)\oplus\mathcal{F}(\mathcal{M}^+)$.

Denote $R:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}\times\mathbb{R}^+$ the mapping $R(z)=\frac{z^2}{|z|}$, where z is the complex number represented as $z=x+iy$. It is clear that R is bi-Lipschitz. Indeed, if $z_0=a+ib$ and $z_1=x+iy$ are two complex numbers from the first quadrant with $a\le x$, then

$$\begin{aligned}|R(z_0)-R(z_1)| &= |e^{a+2ib}-e^{x+2iy}|\le|e^{a+2ib}-e^{a+2iy}|+|e^{a+2iy}-e^{x+2iy}|=\\ &= e^a|e^{ib}-e^{iy}|\cdot|e^{ib}+e^{iy}|+|e^a-e^x|\\ &\le 2|e^{a+ib}-e^{a+iy}|+|e^{a+ib}-e^{x+iy}|\\ &\le 2|e^{a+ib}-e^{x+iy}|+|e^{a+ib}-e^{x+iy}|=3|z_0-z_1|.\end{aligned}$$

On the other hand, for any $z_0=a+ib$ and $z_1=x+iy$ from the upper half plane with $a\le x$ we have

$$\begin{aligned}|R^{-1}(z_0)-R^{-1}(z_1)| &= |e^{a+\frac{ib}{2}}-e^{x+\frac{iy}{2}}|\le|e^{a+\frac{ib}{2}}-e^{a+\frac{iy}{2}}|+|e^{a+\frac{iy}{2}}-e^{x+\frac{iy}{2}}|=\\ &= e^a\frac{|e^{ib}-e^{iy}|}{|e^{\frac{ib}{2}}+e^{\frac{iy}{2}}|}+|e^a-e^x|\\ &\le\frac{\sqrt{2}}{2}|e^{a+ib}-e^{a+iy}|+|e^{a+ib}-e^{x+iy}|\\ &\le 2|z_0-z_1|,\end{aligned}$$

which we wanted to prove.

The mapping

$$T:\mathcal{M}_1\to\mathbb{R}^{n+1}, T(u,x,y)=(u,R(x,y))$$

takes the net $\mathcal{M}_1$ from the set $\mathbb{R}^{n-1}\times\mathbb{R}^+\times\mathbb{R}^+$ in a bi-Lipschitz way to the net $T(\mathcal{M}_1)$ in the set $\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^+$. Hence $\mathcal{F}(\mathcal{M}_1)\cong\mathcal{F}(T(\mathcal{M}_1))$. Since $\mathcal{M}^+=\mathcal{M}_1\cup\mathcal{L}_2\cup\mathcal{M}_2$ is another net in the second set, by Proposition 5 we obtain

$$\mathcal{F}(\mathcal{M}_1)\cong\mathcal{F}(\mathcal{M}^+)$$

Now thanks to the bi-Lipschitz equivalence $\mathcal{M}_1 \cong \mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$,

$$\mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}_{\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2)$$

Since $\mathcal{M}^+$ is bi-Lipschitz equivalent to $\mathcal{M}^+ \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$ we get

$$\mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \oplus \mathcal{F}_{\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \oplus \mathcal{F}_{\mathcal{L}\cup\mathcal{L}_2\cup\mathcal{L}_3}(\mathcal{M}_2 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3) \quad (6)$$

Using (5) and the obvious

$$\mathcal{F}_{\mathcal{L}\cup\mathcal{L}_1\cup\mathcal{L}_2}(\mathcal{M}_1 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2) \cong \mathcal{F}_{\mathcal{L}\cup\mathcal{L}_2\cup\mathcal{L}_3}(\mathcal{M}_2 \cup \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_3)$$

we finally obtain

$$\mathcal{F}(\mathcal{M}_1) \oplus \mathcal{F}(\mathcal{M}_1) \cong \mathcal{F}(\mathcal{M}^+) \cong \mathcal{F}(\mathcal{M}_1)$$

which ends the inductive step and the proof. ∎

Theorem 8. *Let X be a Banach space such that $X \cong Y \oplus X$, where Y is an infinite dimensional Banach space with a Schauder basis. Let $\mathcal{N}$ be a net in X. Then*

$$\mathcal{F}(\mathcal{N}) \cong (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}.$$

Proof. We may assume without loss of generality that the norm of the direct sum $Y \oplus X$ is in fact equal to the maximum norm $Y \oplus_\infty X$. Using Proposition 5 it suffices to prove the result for just one particular net $\mathcal{N}$. Let $M_k \subset kS_X$, $k \in \mathbb{N}$ be a $(1,2)$-net. Then $\mathcal{N} = \cup_{k=1}^{\infty} M_k$ is a $(1,3)$-net in X. Let $\{e_k\}$ be a bi-monotone normalized Schauder basis of Y. Set $Z = (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}$. It is clear that
$Z \cong (\oplus_{j=1}^{\infty} Z)_{\ell_1}$

We will use Pelczynski's decomposition technique to prove the theorem. Since $\mathcal{F}(\mathcal{N})$ is complemented in Z it only remains to prove that $\mathcal{F}(\mathcal{N})$ contains a complemented subspace isomorphic to Z. Let

$$V_n = \{ke_n \oplus x : x \in M_k, k \in \mathbb{N}\} \subset Y \oplus X$$
$$M = \cup_{n=1}^{\infty} V_n$$

The sets V_n, as subsets of the pointed metric space $(Y \oplus X, \|\cdot\|, 0)$, satisfy the assumptions of Proposition 2 and so

$$\mathcal{F}(M) = (\oplus_{n=1}^{\infty} \mathcal{F}(V_n))_{\ell_1} \cong (\oplus_{n=1}^{\infty} \mathcal{F}(\mathcal{N}))_{\ell_1} \cong Z.$$

We extend the set M into a $(1,3)$-net $\mathcal{M}$ in $Y \oplus X$. Because $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$ it suffices to show that $\mathcal{F}(\mathcal{M})$ contains a complemented copy of $\mathcal{F}(M)$. To this end it is enough to find a Lipschitz retraction $R : \mathcal{M} \to M$. Denote by $[a]$ the integer part of $a \in \mathbb{R}$. First let $r : X \to \mathcal{N}$ be a (non-continuous) retraction such that $[\|x\|] \le \|r(x)\| \le \|x\|$, $\|r(x) - x\| \le 4$ and $\|r(x)\| = \|x\|$ provided $\|x\| \in \mathbb{N}$. Let $s : Y \to Y$ be a (non-continuous) retraction defined for $x = \sum_{i=1}^{\infty} x_i e_i$ by

$$s\left(\sum_{i=1}^{\infty} x_i e_i\right) = \begin{cases} de_k & \text{if } x_k > \max\{x_i : i \neq k\} \cup \{0\}, d = \min_{i\neq k}[x_k - x_i] \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

It is easy to see that $\|r(x) - r(y)\| \le 9\|x - y\|$, $\|s(x) - s(y)\| \le 6\|x - y\|$ provided $\|x - y\| \ge 1$ (i.e. they are Lipschitz for large distances). Indeed,

$$\|r(x) - r(y)\| \le \|r(x) - x\| + \|r(y) - y\| + \|x - y\| \le 8 + \|x - y\| \le 9\|x - y\|$$

Assuming $1 \le \|x - y\| \le \lambda$, we get $|x_i - y_i| \le \lambda, i \in \mathbb{N}$. Suppose that $s(x) = de_k$, $s(y) = te_l$. We claim that $d \le 3\lambda$. Indeed, assuming by contradiction that $x_k \ge d + \max\{x_i : i \neq k\} \ge 3\lambda + \max\{x_i : i \neq k\}$ we obtain that $y_k \ge \lambda + \max\{y_i : i \neq k\}$. Hence $k = l$ and $|d - t| \le 2\lambda + 2$. The same argument yields $t \le 3\lambda$, so finally we obtain $\|s(x) - s(y)\| \le 6\lambda$.

Let $R : \mathcal{M} \to M$ is now defined as

$$R(y \oplus x) = \begin{cases} s(y) \oplus r\left(\frac{\|s(y)\|}{\|x\|}x\right) & \text{if } \|x\| > \|s(y)\| > 0 \\ \frac{\|r(x)\|}{\|s(y)\|}s(y) \oplus r(x) & \text{if } \|s(y)\| \ge \|x\| > 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

We claim that R is a retraction onto M. If $y \oplus x \in M$ then clearly $s(y) = y, r(x) = x$, $\|s(y)\| = \|r(x)\|$ and so $R(y \oplus x) = y \oplus x$. Next, observe that $R(y \oplus x) \in M$ holds for every $y \oplus x \in \mathcal{M}$. Indeed, regardless of the case in the definition of R, we see that the first summand of $R(y \oplus x)$ is a non-negative integer multiple of some basis vector e_n in Y. In the first (and third) case it is obvious, in the second case it follows as the norm of $\frac{\|r(x)\|}{\|s(y)\|}s(y)$ is an integer $\|r(x)\|$. The second summand is the result of an application of the retraction r, and its norm equals the norm of the first summand, hence the value of $R(y \oplus x)$ indeed lies in M.

Next, we claim that R is Lipschitz. Recall that $\mathcal{M}$ is a $(1,3)$-net in a Banach space, so it suffices to prove that there exists a $K > 0$ such that $\|R(y_1 \oplus x_1) - R(y \oplus x)\| \le K$ whenever $\|y_1 \oplus x_1 - y \oplus x\| \le D$, for say $D = 8$. This is well-known and easy to see, as every pair of distinct points $p, q \in \mathcal{M}$ can be connected by a straight segment of length $\|p - q\|$, and a sequence of $[\|p - q\|] + 1$ points on this segment of distance (of consecutive elements) at most one. Each of these points has an approximant from $\mathcal{M}$ of distance at most 3, so it clear that there exists a sequence of $[\|p - q\|] + 1$ points in $\mathcal{M}$ of (consecutive) distance at most $D - 1 = 7$, "connecting" the points p, q, and the result follows by a simple summation of the increments of R along the mentioned sequence.

Let us start the proof of Lipschitzness of R by partitioning $\mathcal{M}$ into three disjoint subsets

$$D_1 = \{y \oplus x : \|x\| > \|s(y)\| \ge 20D\},$$

$$D_2 = \{y \oplus x : \|s(y)\| \geq \|x\| \geq 20D\},$$
$$D_3 = \{y \oplus x : \min\{\|s(y)\|, \|x\|\} < 20D\}.$$

The set D_1 (resp. D_2) corresponds to the case 1 (resp. 2) in the definition of R.

Observe that $\|R(y \oplus x)\| \leq \min\{\|y\|, \|x\|\}$ so it suffices to prove the Lipschitzness of R on the set $D_1 \cup D_2$. Moreover, the sets D_1 and D_2 have in a sense a common "boundary" (in the intuitive sense, which is not contained in D_1) consisting of those elements for which $\|x\| = \|s(y)\|$. It is easy to see that for such elements the first two cases in definition of R may be applied with the same result (although formally we are forced to apply the second case). Suppose now that $p \in D_1, q \in D_2$. A similar argument as above using the straight segment connecting p, q (and a short finite sequence from $\mathcal{M}$ which approximates this segment) we see that the segment essentially has to "cross the boundary" between D_1, D_2, and so the proof of the Lipschitzness of R will follow provided we can do it for each of the sets D_1, D_2 separately.

Suppose $y_1 = y + \tilde{y}, x_1 = x + \tilde{x}$ are such that $\|\tilde{y}\|, \|\tilde{x}\| \leq D$.

Case 1. We consider first the case $y_1 \oplus x_1, y \oplus x \in D_1$. Then

$$\frac{\|s(y_1)\|}{\|x_1\|}x_1 - \frac{\|s(y)\|}{\|x\|}x = \frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}(x+\tilde{x}) - \frac{\|s(y)\|}{\|x\|}x =$$
$$\Big(\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\Big)x + \frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}\tilde{x}$$

Now

$$\left|\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\right| \leq \max\left\{\frac{\|s(y)\|+9D}{\|x\|-D} - \frac{\|s(y)\|}{\|x\|}, \frac{\|s(y)\|}{\|x\|} - \frac{\|s(y)\|-9D}{\|x\|+D}\right\}$$

$$\frac{\|s(y)\|+9D}{\|x\|-D} - \frac{\|s(y)\|}{\|x\|} = \frac{(\|s(y)\|+9D)\|x\| - \|s(y)\|(\|x\|-D)}{(\|x\|-D)\|x\|}$$
$$= \frac{9D\|x\| + D\|s(y)\|}{(\|x\|-D)\|x\|} \leq \frac{10D\|x\|}{(\|x\|-D)\|x\|} \leq \frac{10D}{\frac{9}{10}\|x\|} \leq \frac{100D}{9\|x\|} \leq \frac{12D}{\|x\|}$$

Similarly, we obtain

$$\frac{\|s(y)\|}{\|x\|} - \frac{\|s(y)\|-9D}{\|x\|+D} \leq \frac{10D}{\|x\|}.$$

Hence we obtain

$$\left|\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|} - \frac{\|s(y)\|}{\|x\|}\right| \leq \frac{12D}{\|x\|}.$$

The last term is also estimated similarly:

$$\frac{\|s(y+\tilde{y})\|}{\|x+\tilde{x}\|}\|\tilde{x}\| \leq \frac{\|s(y)\|+9D}{\|x\|-D}D \leq \frac{\|x\|+9D}{\|x\|-D}D \leq 3D$$

So combining the above computations we get

$$\left\|\frac{\|s(y_1)\|}{\|x_1\|}x_1 - \frac{\|s(y)\|}{\|x\|}x\right\| \leq 15D$$

So the mapping $y \oplus x \to \frac{\|s(y)\|}{\|x\|}x$ from D_1 to M takes vectors of distance at most D to vectors of distance at most $15D$. It is now clear that R is Lipschitz on D_1.

Case 2. We consider now $y_1 \oplus x_1, y \oplus x \in D_2$, and denote $z = s(y+\tilde{y}) - s(y)$ (recall that $\|z\| \leq 9D$):

$$\frac{\|r(x_1)\|}{\|s(y_1)\|}s(y_1) - \frac{\|r(x)\|}{\|s(y)\|}s(y) = \frac{\|r(x+\tilde{x})\|}{\|s(y+\tilde{y})\|}s(y+\tilde{y}) - \frac{\|r(x)\|}{\|s(y)\|}s(y)$$

Therefore

$$\left\|\frac{\|r(x+\tilde{x})\|}{\|s(y+\tilde{y})\|}s(y+\tilde{y}) - \frac{\|r(x)\|}{\|s(y)\|}s(y)\right\|$$

$$\leq \max\left\{\left\|\frac{\|r(x)\|+9D}{\|s(y)\|-9D}(s(y)+z) - \frac{\|r(x)\|}{\|s(y)\|}s(y)\right\|,\right.$$
$$\left.\left\|\frac{\|r(x)\|-9D}{\|s(y)\|+9D}(s(y)+z) - \frac{\|r(x)\|}{\|s(y)\|}s(y)\right\|\right\}$$

The first term could be rewritten and estimated as follows:

$$\left\|\frac{(\|r(x)\|+9D)\|s(y)\|}{(\|s(y)\|-9D)\|s(y)\|}(s(y)+z) - \frac{(\|s(y)\|-9D)\|r(x)\|}{(\|s(y)\|-9D)\|s(y)\|}s(y)\right\|$$

$$\leq \left\|\left(\frac{(\|r(x)\|+9D)\|s(y)\|}{(\|s(y)\|-9D)\|s(y)\|} - \frac{(\|s(y)\|-9D)\|r(x)\|}{(\|s(y)\|-9D)\|s(y)\|}\right)s(y)\right\| + \frac{\|r(x)\|+9D}{\|s(y)\|-9D}9D$$

$$\leq \left\|\left(\frac{(\|r(x)\|+9D)\|s(y)\| - (\|s(y)\|-9D)\|r(x)\|}{(\|s(y)\|-9D)\|s(y)\|}\right)s(y)\right\| + 27D$$

$$\leq \left|\frac{(\|r(x)\|+9D)\|s(y)\| - (\|s(y)\|-9D)\|r(x)\|}{\|s(y)\|-9D}\right| + 27D$$

$$\leq \left|\frac{9D\|s(y)\|+9D\|r(x)\|}{\|s(y)\|-9D}\right| + 27D \leq \frac{18D\|s(y)\|}{\|s(y)\|-9D} + 27D \leq 63D.$$

The second term we estimate analogously

$$\left\|\frac{\|r(x)\|-9D}{\|s(y)\|+9D}(s(y)+z) - \frac{\|r(x)\|}{\|s(y)\|}s(y)\right\|$$

$$\leq \left\|\left(\frac{(\|r(x)\|-9D)\|s(y)\|}{(\|s(y)\|+9D)\|s(y)\|} - \frac{(\|s(y)\|+9D)\|r(x)\|}{(\|s(y)\|+9D)\|s(y)\|}\right)s(y)\right\| + \frac{\|r(x)\|-9D}{\|s(y)\|+9D}9D$$

$$\leq \left|\frac{9D\|s(y)\|+9D\|r(x)\|}{\|s(y)\|+9D}\right| + 9D \leq \frac{18D\|s(y)\|}{\|s(y)\|} + 9D \leq 27D.$$

We conclude that R is Lipschitz on the whole domain $\mathcal{M}$. Hence $\mathcal{F}(M)$ is isomorphic to a complemented subspace of $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$. ∎

A simple situation which fits the above assumptions is when X contains a complemented subspace with a symmetric basis (e.g. ℓ_p, c_0 or an Orlicz sequence space). By the standard structural theorems for classical Banach spaces ([27]) we obtain.

Corollary 9. *Let X be a Banach space isomorphic to any of the (classical) spaces ℓ_p, L_p, $1 \leq p < \infty, C(K)$, or an Orlicz space h_M, $\mathcal{N}$ be a net in X. Then*

$$\mathcal{F}(\mathcal{N}) \cong (\oplus_{j=1}^{\infty} F(\mathcal{N}))_{\ell_1}.$$

Recall that a metric space M is an absolute Lipschitz retract if, for some $K > 0$, M is a K-Lipschitz retract of every metric superspace $M \subset N$ ([1] p.13). We are going to use the discretized form of this condition. This concept is almost explicit in the work of Kalton [21], where it would probably be called absolute coarse retract.

Definition 3. *Let M be a δ-uniformly discrete space, $\delta > 0$. We say that M is an absolute uniformly discrete Lipschitz retract if, for some $K > 0$, the space M is a K-Lipschitz retract of every δ-uniformly discrete superspace $M \subset N$.*

Lemma 10. *Let X be Banach space which is an absolute Lipschitz retract, $\mathcal{N}$ be a net in X. Then $\mathcal{N}$ is absolute uniformly discrete Lipschitz retract. Conversely, if $\mathcal{N}$ is absolute uniformly discrete Lipschitz retract and X is a Lipschitz retract of X^{**} then X is an absolute Lipschitz retract.*

Proof. The first implication is obvious. To prove the second one, suppose that $X \subset \ell_\infty(\Gamma) = Y$ is a linear embedding. Since $\ell_\infty(\Gamma)$ is an injective space, it suffices to prove that there is a Lipschitz retraction from $\ell_\infty(\Gamma)$ onto X. Since X is a Lipschitz retract of X^{**}, it suffices to follow verbatim the proof of Theorem 1 in [25]. Indeed, consider a net $\mathcal{N}$ in X with extension into a net $\mathcal{M}$ in Y. By assumption, there exists a Lipschitz retraction $r : \mathcal{M} \to \mathcal{N}$. This retraction r can be easily extended to a coarsely continuous retraction R from Y onto X (using the terminology of [21]), which is of course Lipschitz for large distances. It is this condition on R that is used in the proof of Theorem 1 in [25]. ■

Remark. It is an open problem if the retraction from X^{**} to X exists for every separable Banach space (see [21]).

Important examples of absolute uniformly discrete Lipschitz retract are the nets in $C(K)$ spaces, K metrizable compact, [1] p.15.

Corollary 11. *Let $\mathcal{M}$ be a countable absolute uniformly discrete Lipschitz retract which contains a bi-Lipschitz copy of the net $\mathcal{N}$ in c_0. Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$.*

Proof. There is a Lipschitz retraction from $\mathcal{M}$ onto $\mathcal{N}$, and on the other hand using Aharoni's theorem ([11] p. 546) $\mathcal{M}$ is bi-Lipschitz embedded into $\mathcal{N}$ (and hence also a retract). Thus $\mathcal{F}(\mathcal{M})$ is complemented in $\mathcal{F}(\mathcal{N})$ and vice versa. To finish, apply Theorem 8 for c_0 together with the Pelczynski decomposition principle. ■

To give concrete applications of the above corollary, we obtain the following result. The case of c_0^+ follows from the Pelant c_0^+-version of Aharoni's result [30].

Theorem 12. *Let $\mathcal{N}$ be a net in c_0 and $\mathcal{M}$ be a net in any of the following metric spaces: $C(K)$, K infinite metrizable compact, or c_0^+ (the subset of c_0 consisting of elements with non-negative coordinates). Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$.*

3 Schauder basis

Theorem 13. *Let X be a metric space. Suppose there exist a set $M \subseteq X$ and a sequence of distinct points $\{\mu_n\}_{n=1}^{\infty} \subseteq M$, together with a sequence of retractions $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n : M \to M$, $n \in \mathbb{N}$, which satisfy the following conditions:*

(i) $\varphi_n(M) = M_n := \bigcup_{j=1}^{n} \{\mu_j\}$ *for every* $n \in \mathbb{N}$,

(ii) $\overline{\bigcup_{j=1}^{\infty} \{\mu_j\}}^{X} = M$,

(iii) There exists $K > 0$ such that φ_n is K-Lipschitz for every $n \in \mathbb{N}$,

(iv) $\varphi_m \varphi_n = \varphi_n \varphi_m = \varphi_n$ *for every $m, n \in \mathbb{N}$, $n \leq m$.*

Then the space $\mathcal{F}(M)$ has a Schauder basis with the basis constant at most K.

Proof. It is a well-known fact that every Lipschitz mapping $L : A \to B$ between pointed metric spaces A, B, such that $L(0) = 0$ extends uniquely to a linear mapping
$\widehat{L} : \mathcal{F}(A) \to \mathcal{F}(B)$ in a way that that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\widehat{L}} & \mathcal{F}(B) \\ {\scriptstyle \delta_A}\uparrow & & \uparrow{\scriptstyle \delta_B} \\ A & \xrightarrow{L} & B \end{array}$$

Moreover, the norm of $\widehat{L}$ is at most $\mathrm{Lip}(L)$. Therefore for every $n \in \mathbb{N}$ there is a linear mapping $P_n = \widehat{\varphi_n} : \mathcal{F}(M) \to \mathcal{F}(M)$ extending $\varphi_n : M \to M$ with $\|P_n\| \leq K$. We want to prove that $\{P_n\}$ is a sequence of canonical projections associated with some Schauder basis of $\mathcal{F}(M)$, namely that

a) $\dim P_n(\mathcal{F}(M)) = n - 1$ for every $n \in \mathbb{N}$,

b) $P_n P_m = P_m P_n = P_m$ for all $m, n \in \mathbb{N}$, $m \leq n$,

c) $\lim_n P_n(x) = x$ for all $x \in \mathcal{F}(M)$.

The first condition is easy: as $\varphi_n(M) = M_n = \{\mu_i\}_{i=1}^{n}$ we have $P_n(\mathcal{F}(M)) = \mathcal{F}(M_n)$, which is a $(n-1)$-dimensional space. Let us check the commutativity. Note first that for $m, n \in \mathbb{N}$ the diagram

$$\begin{array}{ccccc} \mathcal{F}(M) & \xrightarrow{P_m} & \mathcal{F}(M) & \xrightarrow{P_n} & \mathcal{F}(M) \\ {\scriptstyle \delta_M}\uparrow & & \uparrow{\scriptstyle \delta_M} & & \uparrow{\scriptstyle \delta_M} \\ M & \xrightarrow{\varphi_m} & M & \xrightarrow{\varphi_n} & M \end{array}$$

commutes, which means that $\widehat{\varphi_n \circ \varphi_m} = P_n P_m$. But then from the condition iv follows $P_n P_m = P_m P_n = P_m$ for $m \leq n$.

The validity of the limit equation is proved easily. Note that elements of the form $\sum_{i=1}^{m} \alpha_i \delta_{x_i}$, where $m \in \mathbb{N}$, $x_i \in \{\mu_n\}_{n=1}^{\infty}$, $\alpha_i \in \mathbb{R}$ for all $i \in \{1, \cdots, m\}$, are norm dense in $\mathcal{F}(M)$. Indeed, it is a well-known fact that elements $\mu \in \mathcal{F}(M)$ of the same form $\sum_{i=1}^{m} \alpha_i \delta_{x_i}$ with $x_i \in M$ are norm dense in $\mathcal{F}(M)$ and the condition ii gives the more general result. By uniform boundedness of the family $\{P_n\}_{n=1}^{\infty}$, it suffices to check the limit for elements mentioned above. Thus pick a measure $\mu = \sum_{i=1}^{m} \alpha_i \delta_{x_i}$, $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $x_i \in \{\mu_j\}_{j=1}^{\infty}$ for all $i \in \{1, ..., m\}$. Find $k \in \mathbb{N}$ such that $\{x_1, \cdots, x_m\} \subseteq M_k$. Then for all $n \geq k$ we have

$$\begin{aligned}
||P_n\mu - \mu|| &= \sup_{||f|| \leq 1} \left| \langle f, \sum_{i=1}^{m} \alpha_i (\delta_{\varphi_n(x_i)} - \delta_{x_i}) \rangle \right| \\
&= \sup_{||f|| \leq 1} \left| \sum_{i=1}^{m} (\alpha_i f(\varphi_n(x_i)) - \alpha_i f(x_i)) \right| \\
&= \sup_{||f|| \leq 1} \left| \sum_{i=1}^{m} (\alpha_i f(x_i) - \alpha_i f(x_i)) \right| = 0.
\end{aligned}$$

This was to prove. ∎

Definition 4. *Let X be a Banach space with a Schauder basis $E = \{e_i\}_{i=1}^{\infty}$. The set $M(E) = \{x \in X |\ x = \sum_{i=1}^{\infty} x_i e_i,\ x_i \in \mathbb{Z}, i \in \mathbb{N}\}$ we call the integer-grid to the basis E. If it is clear what basis we are working with, we will denote the set M and speak simply about a grid.*

It is not difficult to see that if a basis E is normalized, then the grid $M(E)$ is a $\frac{1}{2\mathrm{bc}(E)}$-separated set, where $\mathrm{bc}(E)$ denotes the basis constant of E. For E an unconditional basis we will denote $\mathrm{uc}(E)$ the unconditional constant of E. We will now show that for a normalized, unconditional basis E the space $\mathcal{F}(M)$ has a Schauder basis.

Lemma 14. *Let X be a Banach space with a normalized, unconditional Schauder basis $E = \{e_i\}_{i \in \mathbb{N}}$ and a grid $M(E) = M$. Then there exists a sequence of retractions $\varphi_n : M \to M$ together with a sequence of distinct points $\mu_n \in M$, $n \in \mathbb{N}$ satisfying the conditions from the Theorem 13 with the constant at most $K = uc(E) + 2bc(E)$.*

Proof. Before we define the retractions $\{\varphi_n\}_{n=1}^{\infty}$ and the points $\{\mu_n\}_{n=1}^{\infty}$ rigorously, let us give the reader some geometric idea of how will the retractions look like. We will add points from M so that first the set $C_1^1 = \{x_1 e_1 |\ |x_1| \leq 1\}$ is created, then the set $C_1^2 = \{x_1 e_1 + x_2 e_2 |\ |x_i| \leq 1, i = 1, 2\}$, then the set $C_2^2 = \{x_1 e_1 + x_2 e_2 |\ |x_i| \leq 2, i = 1, 2\}$, then $C_2^3 = \left\{\sum_{i=1}^{3} x_i e_i |\ |x_i| \leq 2, i = 1, 2, 3\right\}$ and so on. Note that coordinates of each $\mu \in C_i^j$ are entire numbers.

The retractions will cut coordinates of the argument so that if $x = \sum_{i=1}^{\infty} x_i e_i \in M$ and $\{\mu_i\}_{i=1}^{n} = M_n = \varphi_n(M)$, $n \in \mathbb{N}$, then $\varphi_n(x)$ is obtained by following algorithm: Choose all $\mu_i \in M_n$ minimizing the value $|x_1 - (\mu_i)_1|$, out of them choose those μ_{i_j} minimizing $|x_2 - (\mu_{i_j})_2|$ and so on. Note the process will stop eventually because $x = \sum_{i=1}^{k} x_i e_i$ for some $k \in \mathbb{N}$ as $x \in M$ and the basis E is normalized. It will be a matter of choosing (ordering) the points $\{\mu_i\}_{i=1}^{\infty}$ so that the process ends with only one point $\mu_i = \varphi_n(x)$.

We are now going to describe the construction of the sequence φ_n in the following way. We will build the sequence of points μ_n and to each $n \in \mathbb{N}$, we associate the sets $\varphi_n^{-1}(\mu_i), i \in \{1, \cdots, n\}$. As we want the image $\varphi_n(M) = M_n = \bigcup_{i=1}^{n} \{\mu_i\}$, the only things needed for the mapping φ_n to be well-defined is to check $\bigcup_{i=1}^{n} \{\varphi_n^{-1}(\mu_i)\} = M$ and $\varphi_n^{-1}(\mu_i) \cap \varphi_n^{-1}(\mu_j) = \emptyset$ for $i \neq j$. For simplicity, we denote the set-valued mapping $\varphi_n^{-1} = F_n$ and we will define the mappings $\varphi_n, n \in \mathbb{N}$ through defining $F_n : M_n \to 2^M$. Note that if for every $i \in \{1, ..., n\}$ holds $\mu_i \in F_n(\mu_i)$, then the mapping φ_n is a retraction.

In the sequel, by the n-tuple $(a_1, a_2, ... a_n), a_i \in \mathbb{R}$ we will mean the linear combination $\sum_{i=1}^{n} a_i e_i$ and for a point $x \in X$, $x = \sum_{i=1}^{\infty} x_i e_i$ we will always identify x with $(x_1, x_2, x_3, ...)$.

Set

$$\begin{aligned}
&\mu_1 = 0 && F_1(\mu_1) = M, \\
&\mu_2 = (1,0) && F_2(\mu_2) = \{x \in M |\ x_1 \geq 1\} \\
& && F_2(\mu_1) = M \setminus F_2(\mu_2), \\
&\mu_3 = (-1,0) && F_3(\mu_3) = \{x \in M |\ x_1 \leq -1\} \\
& && F_3(\mu_1) = F_2(\mu_1) \setminus F_3(\mu_3) \\
& && F_3(\mu_2) = F_2(\mu_2).
\end{aligned}$$

It is not difficult to see $\varphi_1, \varphi_2, \varphi_3$ are retractions satisfying the conditions i,iii,iv from the Theorem 13 with Lipschitz constant which equals to $\mathrm{uc}(E) \leq K$. Indeed, for φ_1 it is clear as its image is only $\{0\}$. For φ_2, $x, y \in M$ and $i \in \mathbb{N}$ we have

$$|\varphi_2(x)_i - \varphi_2(y)_i| = \begin{cases} 0 & i > 1 \vee (x_1 \geq 1 \vee y_1 \geq 1) \vee (x_1 \leq 0 \vee y_1 \leq 0), \\ 1 & i = 1 \wedge ((x_1 \geq 1 \wedge y_1 \leq 0) \vee (y_1 \geq 1 \wedge x_1 \leq 0)), \end{cases} \tag{9}$$

and similarily for $n = 3$, $x \in M$ and $i \in \mathbb{N}$ we have

$$\varphi_3(x)_i = \begin{cases} 0 & i > 1 \vee x_1 = 0, \\ 1 & i = 1 \wedge x_1 \geq 1, \\ -1 & i = 1 \wedge x_1 \leq -1 \end{cases}$$

and therefore for $x, y \in M$

$$|\varphi_3(x)_i - \varphi_3(y)_i| = \begin{cases} 0 & i > 1 \vee x_1 y_1 \geq 1 \vee x_1 = y_1 = 0, \\ 1 & i = 1 \wedge ((|x_1| \geq 1 \wedge y_1 = 0) \vee (|y_1| \geq 1 \wedge x_1 = 0)), \\ 2 & i = 1 \wedge x_1 y_1 \leq -1. \end{cases} \tag{10}$$

Due to the unconditionality of E, it is true that for every $x \in X$ and $z \in \mathbb{R}$, $|z| \leq x_1$ holds $\|(z, x_2, x_3, x_4, ...)\| \leq \mathrm{uc}(E)\|x\|$. But for every $i \in \mathbb{N}$ the expression in (10) is less or equal to $|x_i - y_i|$, which gives us Lipschitz condition on φ_n with constant $\mathrm{uc}(E)$.

Moreover, the last retraction φ_3 maps M onto the set $C_1^1 \subseteq M$ containing all points $x \in M$ with $x = (x_1)$ and $|x_1| \leq 1$. Let us denote $C_r^d = \{x \in M | \; x = (x_1, x_2, ..., x_d), |x_i| \leq r, \; i \leq d\}$. From now on, we will proceed inductively. Suppose we have a sequence of retractions $\{\varphi_i\}_{i=1}^m$ together with the points μ_i, such that $\varphi_m(M) = C_r^r$ and that $\{\varphi_i\}_{i=1}^m$ satisfy the conditions i,iii,iv from the Theorem 13. Note that $m = (2r+1)^r$.

We proceed by induction which we divide into two steps. First we find points $\mu_{m+1}, ..., \mu_s$ together with retractions $\varphi_{m+1}, ..., \varphi_s$, where $s = (2r+1)^{r+1}$, such that $M_s = C_r^{r+1}$ and such that $\{\varphi_i\}_{i=1}^s$ satisfy the conditions i,iii,iv from theorem 13. Then we find points $\mu_{s+1}, ..., \mu_t$ and retractions $\varphi_{s+1}, ..., \varphi_t$, where $t = (2r+3)^{r+1}$, $\varphi_t : M \to C_{r+1}^{r+1}$ which satisfy i,iii,iv. As $\bigcup_{r=1}^\infty C_r^r = M$, the condition ii from theorem 13 is obtained as well, which will conclude the proof.

On the bounded set C_r^r we define an ordering by the formula

$$\begin{aligned} (x_1, x_2, ..., x_r) \prec (y_1, y_2, ..., y_r) \Leftrightarrow \; (x_1 > y_1) \vee \\ \exists i \in \{1, ..., r-1\} \, \forall j \in \{1, ..., i\} : (x_j = y_j) \wedge (x_{i+1} > y_{i+1}). \end{aligned} \tag{11}$$

There exists a bijection $w : \{1, ..., (2r+1)^r\} \to C_r^r$, which preserves order.

Let us shorten the notation by introducing indexing functions a, b. If $j \in \{1, ..., r\}$ and $i \in \{1, ..., (2r+1)^r\}$, let $a(j,i) = j(2r+1)^r + i$ and $b(j,i) = (r+j)(2r+1)^r + i$. We set $\mu_{a(j,i)} = (w(i), j) = w(i) + je_{r+1}$ and $\mu_{b(j,i)} = (w(i), -j) = w(i) - je_{r+1}$. Moreover, we formally put $\mu_{a(0,i)} = \mu_{b(0,i)} = w(i)$. Then we define sets

$$\begin{aligned} &F_{a(j,i)}(\mu_{a(j,i)}) = \left\{x \in F_{a(j,i)-1}(\mu_{a(j-1,i)}), x_{r+1} \geq j\right\}, \\ &F_{a(j,i)}(\mu_{a(j-1,i)}) = F_{a(j,i)-1}(\mu_{a(j-1,i)}) \setminus F_{a(j,i)}(\mu_{a(j,i)}), \\ &F_{a(j,i)}(\mu_q) = F_{a(j,i)-1}(\mu_q), \;\; q \in \{1, ..., a(j,i)-1\}, \mu_q \neq \mu_{a(j-1,i)}, \end{aligned}$$

and

$$\begin{aligned} &F_{b(j,i)}(\mu_{b(j,i)}) = \left\{x \in F_{b(j,i)-1}(\mu_{b(j-1,i)}), x_{r+1} \leq -j\right\}, \\ &F_{b(j,i)}(\mu_{b(j-1,i)}) = F_{b(j,i)-1}(\mu_{b(j-1,i)}) \setminus F_{b(j,i)}(\mu_{b(j,i)}), \\ &F_{b(j,i)}(\mu_q) = F_{b(j,i)-1}(\mu_q), \;\; q \in \{1, ..., b(j,i)-1\}, \mu_q \neq \mu_{b(j-1,i)}. \end{aligned}$$

It is easy to see that the formulae above define mappings $\varphi_{a(j,i)}$ and $\varphi_{b(j,i)}$. Supposed it holds for the mappings $\{\varphi_i\}_{i=1}^m$ it is clear that $F_n(\mu_p) \cap F_n(\mu_q) = \emptyset$ for $p \neq q$ and all $n \in \{1, ..., s\}$, and that $\mu_n \in F_n(\mu_n)$ and $\bigcup_{i=1}^n F_n(\mu_i) = M$, which means each mapping φ_n is well-defined and is a retraction onto M_n.

Let us check the uniform Lipschitz boundedness. Fix $n \in \{m+1, ..., s\}$. Note first that

$$\begin{aligned} &\forall x = \sum_{i=1}^\infty x_i e_i \in X, \forall z \in \ell_\infty : \\ &\forall i \in \mathbb{N} : 0 \leq |z_i| \leq |x_i| \Rightarrow \left\| \sum_{i=1}^\infty z_i e_i \right\| \leq \|x\| \cdot \mathrm{uc}(E) \end{aligned}$$

From this we deduce the Lipschitz boundedness.

If $x, y \in M$, then for $i > r+1$ we have $|\varphi_n(x)_i - \varphi_n(y)_i| = |0-0| = 0 \le |x_i - y_i|$. If $i < r+1$ then we distinguish three cases:

a) $|x_i|, |y_i| \le r$. Then $\varphi_n(x)_i = x_i$, $\varphi_n(y)_i = y_i$ and therefore $|\varphi_n(x)_i - \varphi_n(y)_i| = |x_i - y_i|$.

b) $|x_i| \le r$, $|y_i| > r$. Then $\varphi_n(x)_i = x_i$ and $\varphi_n(y)_i = r\,\mathrm{sgn}(y_i)$. Therefore $|\varphi_n(x)_i - \varphi_n(y)_i| = |x_i - r\,\mathrm{sgn}(y_i)| \le |x_i - y_i|$.

c) $|x_i|, |y_i| > r$. Then $\varphi_n(x)_i = r\,\mathrm{sgn}(x_i)$, $\varphi_n(y)_i = r\,\mathrm{sgn}(y_i)$ and therefore

$$|\varphi_n(x)_i - \varphi_n(y)_i| = |r\,\mathrm{sgn}(x_i) - r\,\mathrm{sgn}(y_i)| = \begin{cases} 0 \le |x_i - y_i|, & x_i y_i > 0, \\ 2r \le |x_i - y_i|, & x_i y_i < 0. \end{cases}$$

Finally, let $i = r+1$. If now $x_i y_i < 0$, then either $0 \le \varphi_n(x)_i \le x_i$ and $y_i \le \varphi_n(y)_i \le 0$ or vice versa. Both options give $|\varphi_n(x)_i - \varphi_n(y)_i| \le |x_i - y_i|$, which is what we need.

Let $x_i, y_i \ge 0$. Suppose $n = a(j,k)$ for eligible j,k. Then $\varphi_n(x)_i = j$ or $\varphi_n(x)_i = j-1$ or $\varphi_n(x)_i = x_i$, which occurs whenever $0 \le x_i < j-1$. Of course the same holds for y. From this we have either $|\varphi_n(x)_i - \varphi(y)_i| \le |x_i - y_i|$ or $|\varphi_n(x)_i - \varphi(y)_i| \le 1$. If $n = b(j,k)$ for some j,k, then $\varphi_n(x)_i = r$ whenever $x_i \ge r$ and $\varphi_n(x)_i = x_i$ whenever $x_i < r$, the same for y. It is clear that $|\varphi_n(x)_i - \varphi(y)_i| \le |x_i - y_i|$.

Let $x_i, y_i \le 0$. If $n = a(j,k)$ for some j,k, then $|\varphi_n(x)_i - \varphi(y)_i| = |0-0| = 0 \le |x_i - y_i|$. If $n = b(j,k)$ for some j,k, then $\varphi_n(x)_i = -j$ or $\varphi_n(x)_i = -j+1$ or $\varphi_n(x)_i = x_i$, which holds whenever $0 \ge x_i > -j+1$. Again, we get either $|\varphi_n(x)_i - \varphi(y)_i| \le |x_i - y_i|$ or $|\varphi_n(x)_i - \varphi(y)_i| \le 1$.

To sum up all cases, if $x, y \in M$, then either $x_{r+1} = y_{r+1}$ or not. In the first case we have

$$\begin{aligned} \|\varphi_n(x) - \varphi_n(y)\| &= \left\| \sum_{i=1}^{r+1} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| \le \left\| \sum_{i=1}^{r} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| + 1 \\ &\le \mathrm{uc}(E)\|x-y\| + 2\mathrm{bc}(E)\|x-y\| = \\ &= \|x-y\|(\mathrm{uc}(E) + 2\mathrm{bc}(E)), \end{aligned} \tag{12}$$

as M is a $\frac{1}{2\mathrm{bc}(E)}$-separated set, while in the $x_{r+1} \ne y_{r+1}$ case we have

$$\|\varphi_n(x) - \varphi_n(y)\| = \left\| \sum_{i=1}^{r+1} (\varphi_n(x)_i - \varphi_n(y)_i)\, e_i \right\| \le \mathrm{uc}(E)\|x-y\|. \tag{13}$$

Considering both cases we get the mapping φ_n is Lipschitz with constant $K = \mathrm{uc}(E) + 2\mathrm{bc}(E)$.

It remains to prove that the mappings $\{\varphi_n\}_{n=1}^{s}$ satisfy the commutativity condition iv, provided the mappings $\{\varphi_n\}_{n=1}^{m}$ do. Note that for any $m, n \in \mathbb{N}$, $m \le n$ holds

$$F_n(\mu_n) \cap F_m(\mu_m) \in \{\emptyset, F_n(\mu_n)\}\,. \tag{14}$$

Out of this fact the commutativity follows easily: Consider $i < j \in \{1, ..., s\}$. First, because φ_i is a retraction onto M_i and the same holds for φ_j and M_j, from $M_i \subseteq M_j$ follows $\varphi_j \varphi_i = \varphi_i$. It remains to prove $\varphi_i \varphi_j(x) = \varphi_i(x)$ for every $x \in M$.

Take $x \in M$. There exists a maximal finite sequence of indices $1 = k_0 < ... < k_l \leq s$ such that

$$x \in F_{k_l}(\mu_{k_l}) \subseteq \cdots \subseteq F_{k_0}(\mu_{k_0}).$$

Clearly if $c(i)$ is the biggest index such that $k_{c(i)} \leq i$, then $\varphi_i(F_{k_d}(\mu_{k_d})) = \mu_{c(i)}$ for all d, $c(i) \leq d \leq l$. This applies analogously for φ_j with $c(j)$. From the fact that both $x, \mu_{k_{c(j)}} \in F_{k_{c(j)}}(\mu_{k_{c(j)}}) \subseteq F_{k_{c(i)}}(\mu_{k_{c(i)}})$ we get simply

$$\varphi_i \varphi_j(x) = \varphi_i(\mu_{k_{c(j)}}) = \mu_{k_{c(i)}} = \varphi_i(x),$$

which finishes the proof of commutativity.

To finish the proof, it remains to show the construction of retractions $\varphi_{s+1}, ..., \varphi_t$, where $t = (2r+3)^{r+1}$, $\varphi_t : M \to C_{r+1}^{r+1}$ which satisfy i,iii,iv.

For $i \in \mathbb{N}$ let us define an i-predecessor function $p_i : M \to M$ by

$$p_i\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n e_n - \operatorname{sgn}(x_i) e_i.$$

Now for every $j \in \{1, ..., r+1\}$ we introduce sets

$$\begin{aligned} A_{j,1} = \{&(x_1, ..., x_{j-1}, r+1, x_{j+1}, ..., x_{r+1}) : \\ &x_i \in \mathbb{Z} \wedge |x_i| \leq r+1 \text{ for } i < j \wedge |x_i| \leq r \text{ for } i > j\}, \\ A_{j,-1} = \{&(x_1, ..., x_{j-1}, -r-1, x_{j+1}, ..., x_{r+1}) : \\ &x_i \in \mathbb{Z} \wedge |x_i| \leq r+1 \text{ for } i < j \wedge |x_i| \leq r \text{ for } i > j\}. \end{aligned}$$

Clearly, $A_{j,-1}, A_{j,1} \subseteq C_{r+1}^{r+1}$ and $|A_{j,-1}| = |A_{j,1}| = (2r+1)^{r+1-j}(2r+3)^{j-1}$. Moreover,

$$\bigcup_{\substack{j \in \{1, ..., r+1\} \\ i \in \{-1, 1\}}} A_{j,i} = C_{r+1}^{r+1} \setminus C_r^{r+1}$$

and it is a disjoint union. For each j, choose any bijection $w_j : \{1, ..., |A_{j,1}|\} \to A_{j,1}$ and fix it. Define $\overline{w}_j : \{1, ..., |A_{j,1}|\} \to A_{j,-1}$, by $\overline{w}_j(i) = (w_j(i)_1, w_j(i)_2, ..., -w_j(i)_j, ..., w_j(i)_{r+1})$. For simplicity, for $j \in \{1, ..., r+1\}, i \in \{1, ..., |A_{j,1}|\}$ put

$$\alpha(j,i) = s + 2\sum_{k=1}^{j-1} |A_{k,1}| + i, \quad \beta(j,i) = s + 2\sum_{k=1}^{j-1} |A_{k,1}| + |A_{j,1}| + i.$$

Then we finally set $\mu_{\alpha(j,i)} = w_j(i)$, $\mu_{\beta(j,i)} = \overline{w}_j(i)$. Now we define mappings $\{F_n\}_{n=s+1}^{t}$ via

$$\begin{aligned} &F_{\alpha(j,i)}(\mu_{\alpha(j,i)}) = \left\{x \in F_{\alpha(j,i)-1}(p_j(\mu_{\alpha(j,i)})), x_j \geq r+1\right\}, \\ &F_{\alpha(j,i)}(p_j(\mu_{\alpha(j,i)})) = F_{\alpha(j,i)-1}(p_j(\mu_{\alpha(j,i)})) \setminus F_{\alpha(j,i)}(\mu_{\alpha(j,i)}), \\ &F_{\alpha(j,i)}(\mu_q) = F_{\alpha(j,i)-1}(\mu_q), \quad q \in \{1, ..., \alpha(j,i)-1\}, \mu_q \neq p_j(\mu_{\alpha(j,i)}), \end{aligned}$$

and

$$F_{\beta(j,i)}(\mu_{\beta(j,i)}) = \left\{x \in F_{\beta(j,i)-1}(p_j(\mu_{\beta(j,i)})), x_j \leq -r-1\right\},$$
$$F_{\beta(j,i)}(p_j(\mu_{\beta(j,i)})) = F_{\beta(j,i)-1}(p_j(\mu_{\beta(j,i)})) \setminus F_{\beta(j,i)}(\mu_{\beta(j,i)}),$$
$$F_{\beta(j,i)}(\mu_q) = F_{\beta(j,i)-1}(\mu_q),\ \ q \in \{1,...,\beta(j,i)-1\}, \mu_q \neq p_j(\mu_{\beta(j,i)}).$$

Obviously, the upper equations define mappings $\varphi_{\alpha(j,i)}$ and $\varphi_{\beta(j,i)}$ for all $j \in \{1,...,r+1\}$ and $i \in \{1,...,|A_{j,1}|\}$, hence the mappings $\{\varphi_n\}_{n=s+1}^{t}$ are well-defined and it is an easy check that each such φ_n is a retraction onto the set M_n.

Note that the sets $\{F_n(\mu_n)\}_{n=1}^{t}$ still satisfy the condition (14) so the commutativity condition iv from theorem 13 is obtained similarly as it was done for retractions $\{\varphi_n\}_{n=1}^{s}$.

It remains to show the mappings are Lipschitz-bounded. Let us for simplicity denote $\beta_k = \beta(k-1, |A_{k-1,1}|)$ for $1 < k \leq r+1$ and $\beta_1 = s$, the index of first retraction φ_{β_k} such that $A_{k-1,-1} \subseteq M_{\beta_k}$. Fix $n \in \{s+1,...,t\}$. We will prove that there exists at most one $j = j(n) \in \mathbb{N}$ such that for all $l \in \mathbb{N}, l \neq j$ and all $x, y \in M$ we have $|\varphi_n(x)_l - \varphi_n(y)_l| \leq |x_l - y_l|$ out of which the Lipschitz boundedness of φ_n follows. If $n = \alpha(j,i)$ for some eligible j, i, then for every $x \in M$ holds

$$\varphi_n(x)_l = \begin{cases} 0 & l > r+1, \\ x_l & (l \leq r+1, |x_l| \leq r) \vee (l < j, |x_l| = r+1), \\ r\,\mathrm{sgn}(x_l) & (j < l \leq r+1, |x_l| > r) \vee (j = l, x_l < -r) \vee \\ & \left(j = l, x_l > r, \forall \mu \in M_n : \varphi_{\beta_j}(x)_j \neq p_j(\mu)\right), \\ (r+1)\,\mathrm{sgn}(x_l) & (l < j, |x_l| > r+1) \vee \\ & \left(l = j, x_l \geq r+1, \exists \mu \in M_n : \varphi_{\beta_j}(x)_j = p_j(\mu)\right), \end{cases}$$

while if $n = \beta(j,i)$ for some j, i, then for every $x \in M$ we have

$$\varphi_n(x)_l = \begin{cases} 0 & l > r+1, \\ x_l & (l \leq r+1, |x_l| \leq r) \vee (l < j, |x_l| = r+1) \vee \\ & (l = j, x_l = r+1), \\ r\,\mathrm{sgn}(x_l) & (j < l \leq r+1, |x_l| > r) \vee \\ & \left(j = l, x_l < -r, \forall \mu \in M_n : \varphi_{\beta_j}(x)_j \neq p_j(\mu)\right), \\ (r+1)\,\mathrm{sgn}(x_l) & (l < j, |x_l| > r+1) \vee (l = j, x_l > r+1) \vee \\ & \left(l = j, x_l \leq -r-1, \exists \mu \in M_n : \varphi_{\beta_j}(x)_j = p_j(\mu)\right). \end{cases}$$

If $x, y \in M$, it is not difficult to see that if $|\varphi_n(x)_l - \varphi_n(y)_l| > |x_l - y_l|$, then $l = j$ and $\varphi_n(x)_l = (r+1)\,\mathrm{sgn}(x_l)$, $\varphi_n(y)_l = r\,\mathrm{sgn}(y_l)$ or vice versa and $x_l y_l > 0$. Particularly $|\varphi_n(x)_l - \varphi_n(y)_l| = 1$ and $|x_l - y_l| = 0$. For all other l, i.e. $l \neq j, l \in \mathbb{N}$ holds $|\varphi_n(x)_l - \varphi_n(y)_l| \leq |x_l - y_l|$, which is what we need.

Therefore we get by computation similar to those done in (12) and (13) that φ_n is a Lipschitz mapping with constant $K = \mathrm{uc}(E) + 2\mathrm{bc}(E)$, which concludes the induction.

As $\bigcup_{r=1}^{\infty} C_r^r = M$ the condition ii from theorem 13 is also satisfied and hence our proof is finished. ∎

Remark. In (11) it was not necessary for our construction to choose exactly this order. In fact, any bijection $w : \{1, \ldots, (2r+1)^r\} \to C_r^r$ would suit our purpose. We chose the order (11) for simplicity. In this case we have $\mu_{a(j-1,i)} = p_j(\mu_{a(j,i)})$ and $\mu_{b(j-1,i)} = p_j(\mu_{b(j,i)})$ for p_j the j-predecessor function and $i \in \{1, \ldots, (2r+1)^r\}$, $j \in \{1, \ldots, r\}$.

Corollary 15. *If $E = \{e_i\}_{i=1}^{\infty}$ denotes the canonical basis in c_0 and $M = M(E) \subseteq c_0$ the integer grid, then the Free-space $\mathcal{F}(M)$ has a monotone Schauder basis.*

Proof. applying the construction of the retractions from the lemma 14 to (c_0, E), we get Lipschitz constant $K = 1$, (see estimates (12) and (13)). Therefore, $\mathcal{F}(M)$ has a monotone Schauder basis. ∎

Corollary 16. *Let $\mathcal{N} \subseteq c_0$ be a net. Then the Free-space $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. If we use the notation from previous corollary, M is a $(1,1)$-net in c_0. But as all nets in an infinite-dimensional space are Lipschitz equivalent ([1], p.239, Proposition 10.22), $\mathcal{N}$ is Lipschitz equivalent to the grid M and therefore $\mathcal{F}(\mathcal{N})$ is isomorphic to $\mathcal{F}(M)$, which concludes the proof. ∎

Corollary 17. *Let $\mathcal{N}$ be a net in any of the following metric spaces: $C(K)$, K metrizable compact, or c_0^+ (the subset of c_0 consisting of elements with non-negative coordinates). Then $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. Follows immediately from Theorem 12. ∎

Corollary 18. *Let $\mathcal{N} \subseteq \mathbb{R}^n$ be a net. Then $\mathcal{F}(\mathcal{N})$ has a Schauder basis.*

Proof. It follows from the proof of lemma 14 that $\mathcal{F}(\mathbb{Z}^n)$ has a Schauder basis and $\mathcal{F}(\mathbb{Z}^n) \cong \mathcal{F}(\mathcal{N})$ by Proposition 5, which gives the result. ∎

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DISTORTION OF LIPSCHITZ FUNCTIONS ON $c_0(\Gamma)$

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ABSTRACT. Let Γ be an uncountable cardinal. We construct a real symmetric 1-Lipschitz function on the unit sphere of $c_0(\Gamma)$ whose restriction to any nonseparable subspace is a distortion.

1. Introduction

Let us start by recalling the classical definitions of oscillation stability and distortion. Let X be a real infinite dimensional Banach space, and let $f : S_X \to \mathbb{R}$ be a real valued function. The function f is said to be oscillation stable if for every infinite dimensional subspace $Z \subset X$ and $\varepsilon > 0$ there exists a further subspace $Y \subset Z$ such that the oscillation of f on S_Y is at most ε, i.e., $|f(x) - f(y)| \leq \varepsilon$, $x, y \in S_Y$.

The function f is said to be a distortion if there exists an $\varepsilon > 0$ such that for every infinite dimensional subspace Y of X there exist $x, y \in S_Y$ such that $|f(x) - f(y)| \geq \varepsilon$.

It is clear that oscillation stability and distortion are in a sense opposite properties. More precisely, any function f on S_X is either oscillation stable, or it is a distortion on $S_X \cap Y$ for some subspace $Y \subset X$. On the other hand, the distortion passes to subspaces and so a distorting function is not oscillation stable on any subspace of X.

It is a classical result of James [7] that every equivalent norm on the Banach space c_0, resp. ℓ_1, is oscillation stable. On the other hand, the spaces $\ell_p, 1 < p < \infty$, admit a distorting renorming by the results of Odell and Schlumprecht [11]. It turns out, by combining the result of [11] with the work of Milman [10] that every equivalent norm on a Banach space is oscillation stable if and only if the space in question is saturated by copies of c_0, or ℓ_1.

The supply of Lipschitz functions on a Banach space is much larger than that of renormings, so one would expect that distorting Lipschitz functions are more abundant. Using the concepts of asymptotic set ([15], [4], [11]) and the Mazur map, one can transfer the distorting norm from the unit sphere of ℓ_2 into a distorting Lipschitz function on the unit sphere of ℓ_1. So while all equivalent norms on the space ℓ_1 are oscillation stable, Lipschitz functions may be distorting. The details of this procedure are described, e.g., in the article of Odell-Schlumprecht in [12].

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It is important to note that the transfer preserves the symmetry of the mappings involved.

As regards the remaining relevant space c_0, there is the following result by Gowers [3].

Theorem 1 (Gowers)**.** *Every Lipschitz function $f : S_{c_0} \to \mathbb{R}$ is oscillation stable.*

Putting the above-mentioned results together, one can conclude that every Lipschitz function on a Banach space is oscillation stable if and only if the space in question is saturated by copies of c_0.

Our interest in the present note lies in the nonseparable oscillation stabilization (resp. distortion) of Lipschitz functions. More precisely, let X be a nonseparable real Banach space with density character Γ, and $f : S_X \to \mathbb{R}$ be a Lipschitz function. Given a nonseparable subspace Y of X (say of the same density character Γ), and $\varepsilon > 0$ is there a further infinite dimensional subspace Z (of the density character Γ) of Y such that f on S_Z has oscillation at most ε? In the special case of $\ell_p(\Gamma)$ spaces this problem can be resolved by using the separable results combined with their symmetry. Indeed, let $(X, \|\cdot\|)$ be a Banach space with a symmetric (possibly uncountable) Schauder basis $\{e_\gamma\}_{\gamma\in\Gamma}$, where Γ is any nonempty set. We say that a function $f : X \to \mathbb{R}$ is symmetric if the value $f(x)$ is preserved under any permutation of the coordinates of x. It is clear that a symmetric function on X is uniquely determined by its values on the span of any countably infinite set $\{e_{\gamma_i}\}_{i=1}^{\infty}$. Thanks to the construction of Maurey [9] of a distorting and symmetric norm on every $\ell_p, 1 < p < \infty$, it is easy to formally extend the distorting (and symmetric) norms onto $\ell_p(\Gamma), 1 < p < \infty$, for every infinite set Γ. It is immediate to check that the extensions will preserve the distortion.

Similarly, one can extend the distorting Lipschitz and symmetric function from ℓ_1 (constructed using the symmetric distorting norm on ℓ_2), onto arbitrary $\ell_1(\Gamma)$. The distortion property will again be preserved.

It is natural to ask if there exists any nonseparable Banach space X such that all norms (resp. Lipschitz functions) on X are oscillation stable (resp. distorted) in the nonseparable sense. The obvious remaining test space is of course the space $c_0(\Gamma)$.

Our main result, solving the Problem 199 in the recent book of Guirao, Montesinos and Zizler [5], is that there exists a nonseparably distorting Lipschitz function on $c_0(\Gamma)$. More precisely, we have the next result.

Theorem 2. *There is a 1-Lipschitz symmetric function $F : S_{c_0(\Gamma)} \to \mathbb{R}$, such that for every nonseparable subspace $Y \subseteq c_0(\Gamma)$ there are points $x, y \in S_Y$ such that $|F(x) - F(y)| > \frac{1}{4}$.*

On the other hand, the nonseparable oscillation stability of equivalent norms on $c_0(\Gamma)$, resp. $\ell_1(\Gamma)$ still holds. This folklore result is apparently well-known to experts in the field. We would like to thank Tomasz Kania for bringing this fact to our attention. The case of $\ell_1(\Gamma)$ was dealt with in the paper of Giesy [2]. The case of $c_0(\Gamma)$ seems not to have been published in the refereed journal, although there exists a short note of Granero containing the proof. For the convenience of the reader, we have included in this note the formal statement and the proof, which goes along the lines of the classical James theorem.

Theorem 3. *Let Γ be an uncountable cardinal, $X = c_0(\Gamma)$ (resp. $\ell_1(\Gamma)$). For every equivalent norm $|||\cdot|||$ on X, $\varepsilon > 0$ and a subspace $Z \subset X$ there exist a constant $c > 0$ and a subspace $Y \subseteq Z$ with $\operatorname{dens} Y = \operatorname{dens} Z$ such that $c - \varepsilon < |||x||| \le c$ for every point $x \in Y$, $\|x\| = 1$.*

In the sequel we will need the following well-known fact [1, p. 12]. Suppose (M, d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$. Then the following formula defines a K-Lipschitz function $\hat{g} : M \to \mathbb{R}$ such that $\hat{g}|_S = g$:

$$\hat{g}(x) = \inf_{y \in S}\{g(y) + Kd(x, y)\}. \tag{1}$$

In the construction of F, we will use a simple modification of the formula (1), which will ensure that the range of F is contained in $[0, 1]$. We omit the completely straightforward proof.

Lemma 4 (Modified extension formula)**.** *Suppose (M, d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$, taking values only in the interval $[0, 1]$. Then the following formula defines a K-Lipschitz function $\overline{g} : M \to \mathbb{R}$, taking values only in $[0, 1]$ such that $\overline{g}|_S = g$:*

$$\overline{g}(x) = \min\{\inf_{y \in S}\{g(y) + Kd(x, y)\}, 1\}. \tag{2}$$

2. Proofs of the results

Proof of Theorem 2. To prove the theorem, it suffices to construct (as we will do) the symmetric 1-Lipschitz function $F : c_0(\omega_1) \to \mathbb{R}$ and show it does not stabilize on the sphere of any subspace $Y \subseteq c_0(\omega_1)$ with $\operatorname{dens} Y = \omega_1$. Indeed, in the general case we use the symmetric extension of F to $c_0(\Gamma)$, and we check easily that any nonseparable space $Y \subset c_0(\Gamma)$ contains a further nonseparable subspace of density ω_1, which is contained in some $c_0(\Lambda)$, $\Lambda \subset \Gamma$, $|\Lambda| = \omega_1$.

The meaning of symmetry is that the function value $F(x)$ does not depend on the particular distribution of the coordinates of the vector x in the domain, but only on the set of the coordinate values of x. We define an equivalence relation $\sim$ on $c_{00}(\omega_1)$ in the following way: $x \sim y$ whenever $|\operatorname{supp} x| = |\operatorname{supp} y|$ and there exists a bijection f from $\operatorname{supp} x$ to $\operatorname{supp} y$ (both understood as finite sets of ordinal numbers) such that $x(\gamma) = y(f(\gamma))$. We will call every equivalence class $[x] \in X := c_{00}(\omega_1)/\sim$ a shape.

Note that if $x \sim y$, $x, y \in c_{00}(\omega_1)$, then $\|x\| = \|y\|$. Next, let us denote by $L = \{S_i\}_{i=1}^{\infty}$ the sequence of all shapes of norm one with finite support and rational coordinates.

Lemma 5. *Let $x, y, \in S_j, j \in \mathbb{N}$. Then for any shape $S \in L$, $d(x, S) = d(y, S)$ holds.*

Proof. The statement of the lemma follows readily from the fact that for every $w \in S$ there exists a $z \in S$ such that $\|x - w\| = \|y - z\|$. To prove the latter statement, if $x \sim y$, then there is a bijection $\varphi : \operatorname{supp} x \to \operatorname{supp} y$ such that for every $\gamma \in \omega_1$ we have $x(\gamma) = y(\varphi(\gamma))$. Set $s_y = \inf\{\alpha | \ \forall \beta \in \operatorname{supp} y : \beta < \alpha\}$. Define a mapping $\psi : \operatorname{supp} w \to \omega_1$ by

$$\psi(\gamma) = \begin{cases} \varphi(\gamma)\,, & \gamma \in \operatorname{supp} x \cap \operatorname{supp} w\,, \\ \alpha + \gamma\,, & \gamma \in \operatorname{supp} w \setminus \operatorname{supp} x\,. \end{cases}$$

Clearly ψ is a bijection onto its image. If we define z as $z(\psi(\gamma)) = w(\gamma)$ for $\gamma \in \operatorname{supp} w$ and 0 elsewhere, then $z \in [w] = S$. It follows from the definition of z that $\|x - w\| = \|y - z\|$. Therefore $d(x, S) = d(y, S)$ for any shape $S \in L$. □

We define inductively a function $\pi : L \to L$, which is going to "clone" every shape to an identical shape repeated several times in a row. Suppose $x \in S_1$ with $\operatorname{supp} x = \{1, ..., k\}$ for some $k \in \mathbb{N}$. Define $\pi(S_1) = [y]$, where $y(i) = y(k + i) = y(2k + i) = ... = y(k^2 + i) = x(i)$ for $i \in \{1, ..., k\}$ and $y(\gamma) = 0$ for $\gamma \in \omega_1 \setminus \{1, ..., k(k + 1)\}$.

Suppose π has been defined for all S_i, $i < j$, and

$$k = \max\{\max_{i \leq j} |\operatorname{supp} S_i|, \max_{i<j} |\operatorname{supp} \pi(S_i)|\}.$$

If $x \in S_j$ is such that $\operatorname{supp} x = \{1, \ldots, l\}$ for some $l \in \mathbb{N}$, then we set $\pi(S_j) = [y]$, where $y(i) = y(l + i) = y(2l + i) = \cdots = y(kl + i) = x(i)$ for $i \in \{1, \ldots, l\}$ and $y(\gamma) = 0$ for $\gamma \in \omega_1 \setminus \{1, \ldots, (k + 1)l\}$.

Note that for all $i < j \in \mathbb{N}$ the distance between any $x \in \pi(S_j)$ and $y \in S_i \cup \pi(S_i) \cup S_j$ is equal to 1. Indeed, every element x with S_j has some coordinate which equals 1 or -1 and therefore $\pi(S_j)$ has at least $k+1$ such coordinates, where k is the maximum "length" of a support of previously treated shapes (S_i or $\pi(S_i)$ for $i \leq j$ or $i < j$ respectively). Therefore, there exists a point $\gamma \in \omega_1$ where $|x(\gamma)| = 1$ and $y(\gamma) = 0$.

We construct our function F on the set $S = \bigcup\{x : x \in [x], [x] \in L\}$ by an inductive repetition of the extension operation. The extension onto the unit sphere $S_{c_0(\omega_1)}$ is then unique and 1-Lipschitz as the set S is dense in $S_{c_0(\omega_1)}$. Moreover, as the values of F will depend only on the shape $[x] \in L$, it follows that F is symmetric.

Set $F(x) = 0$ for all $x \in S_1$ and $F(y) = 1$ for all $y \in \pi(S_1)$. Such a function is clearly 1-Lipschitz. After having defined F on the set $S_1 \cup \pi(S_1)$, we extend F to the set $S_1 \cup \pi(S_1) \cup S_2$ via the extension formula (2). Of course, if $\pi(S_1) = S_2$, the extension is trivial (as the domain has not increased) and we move to the definition of $F(\pi(S_2))$ described below. Note that $F(S_2) \subseteq [0, 1]$. We will check below in the general inductive step that F is constant on the set S_2. Furthermore, we set $F(\pi(S_2)) = 1$ if $F(S_2) \leq \frac{1}{2}$ and $F(\pi(S_2)) = 0$ if $F(S_2) > \frac{1}{2}$. Thus F is still 1-Lipschitz, as $\pi(S_2)$ has distance one from each of the sets S_1, $\pi(S_1)$ and S_2.

Let us describe the general inductive step. Suppose F has been defined on the sets $S_1, \ldots, S_{j-1}, \pi(S_1), \ldots, \pi(S_{j-1})$ and it is constant on every such a set. We use the formula (2) to extend F to the set S_j if it hasn't been defined there yet. Note that again $F(S_j) \subseteq [0, 1]$. Let us check that F is constant on S_j (or S_l). Pick two points $x, y, \in S_j$. Using (1)

$$\hat{F}(y) = \inf_{w \in \bigcup_{i=1}^{j-1} S_i \cup \pi(S_i)} \{F(w) + \|y - w\|\}. \tag{3}$$

Since by our inductive assumption F is constant on every set S_i, $\pi(S_i)$, $i \in \{1, ..., j - 1\}$, replacing y with x in the formula (3) and using Lemma 5 gives the same value. As the formula gives the same values for all $x \in S_j$, so does the formula (2). We conclude F is constant on S_j.

Finally, having defined F on the sets $S_1, ..., S_j$ and $\pi(S_1), ..., \pi(S_{j-1})$, we set $F(\pi(S_j)) = 1$ if $F(S_j) \leq \frac{1}{2}$ and $F(\pi(S_j)) = 0$ if $F(S_j) > \frac{1}{2}$. We finish the definition of F by extending it continuously onto $S_{c_0(\omega_1)}$. Clearly, F is 1-Lipschitz and symmetric.

Next we are going to show that for every subspace $Y \subseteq c_0(\omega_1)$ with $\operatorname{dens} Y = \omega_1$ there exist two points $x, y \in S_Y$ with $|F(x) - F(y)| > \frac{1}{4}$.

Our next lemma, which is probably a folklore result, is a variant to some results of Rodriquez-Salinas [13].

Lemma 6. *Let $Y \subseteq c_0(\omega_1)$ be a subspace with $\operatorname{dens} Y = \omega_1$. Then there exists a transfinite sequence $\{x_\gamma\}_{\gamma=1}^{\omega_1}$ of norm one vectors from Y with pairwise disjoint supports, i.e.,*

$$\operatorname{supp}(x_\alpha) \cap \operatorname{supp}(x_\beta) = \emptyset, \quad \alpha \neq \beta.$$

In particular, Y contains a subspace isomorphic to $c_0(\omega_1)$.

Proof. We proceed by transfinite induction. Choose a norm one vector $x_1 \in S_Y$. After having chosen $\{x_\gamma : 1 \leq \gamma < \Omega\}$, for some $\Omega < \omega_1$, we consider the countable set $\Lambda = \bigcup_{\gamma<\Omega} \operatorname{supp}(x_\gamma) \subset [1, \omega_1)$. Since $Y \subset c_0(\omega)$ is a nonseparable Weakly Compactly Generated (WCG) space ([6, p. 211]), it is also a Weakly Lindeloff Determined (WLD) space, and so $w^* - \operatorname{dens} Y^* = \omega_1$ ([6, p. 181]). Hence $V = \overline{\{\delta_\gamma : \gamma \in \Lambda\}}^{w^*}$ is the proper w^*-closed subspace of Y^*. Hence $Z = \{y \in Y : y(\gamma) = 0, \ \ \gamma \in \Lambda\} = V_\perp$ is a nontrivial subspace of Y, and we may find the next element of the sequence $x_\Omega \in S_Z$. This procedure yields the desired long sequence $\{x_\gamma : 1 \leq \gamma < \omega_1\}$, which is equivalent to the long Schauder basis of $c_0(\omega_1)$. □

Choose a sequence $\{y_\gamma\}_{\gamma\in\omega_1}$ of norm one vectors in $c_0(\omega_1)$ with finite support and rational coordinates such that $\operatorname{supp} y_\gamma \subseteq \operatorname{supp} x_\gamma$ and $\|x_\gamma - y_\gamma\| < \frac{1}{8}$. As $\{y_\gamma\}_{\gamma\in\omega_1}$ is an uncountable sequence and L is countable, it follows that there exists a shape $S \in L$ which corresponds to infinitely many y_γ. So there is an infinite sequence of distinct indices $\{\gamma_i\}_{i=1}^{\infty}$ from the set ω_1 such that $y_{\gamma_i} \in S$ for each $i \in \mathbb{N}$. Let d be the number of times S is cloned in $\pi(S)$. Then set

$$x = \sum_{i=1}^{d} x_{\gamma_i}, \ y = \sum_{i=1}^{d} y_{\gamma_i}$$

and observe $x \in Y$, $\|x - y\| < \frac{1}{8}$. Indeed,

$$\|x - y\| = \sup_{\alpha\in\omega_1} \left| \sum_{i=1}^{d} x_{\gamma_i}(\alpha) - \sum_{i=1}^{d} y_{\gamma_i}(\alpha) \right| = \max_{i\in\{1,\ldots,d\}} \sup\nolimits_{\alpha\in\operatorname{supp} x_{\gamma_i}} |x_{\gamma_i}(\alpha) - y_{\gamma_i}(\alpha)|$$

$$= \max_{i\in\{1,\ldots,d\}} \|x_{\gamma_i} - y_{\gamma_i}\| < \frac{1}{8}$$

as all the x_{γ_i} have disjoint supports and $\operatorname{supp} y_{\gamma_i} \subseteq \operatorname{supp} x_{\gamma_i}$, $i \in \{1, ..., d\}$. Therefore we get

$$|F(x) - F(x_{\gamma_1})| \geq |F(y) - F(y_{\gamma_1})| - |F(x) - F(y)| - \|x_{\gamma_1} - y_{\gamma_1}\| \geq \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4}.$$

□

The strategy of the proof of Theorem 3 is similar to that of the classical James proof in the separable case. Namely, we are constructing a (long) sequence of disjointly supported vectors in Z (equivalent to the canonical basis of X) so that the one-sided estimate of the norm on their linear span nearly satisfies either the supremum (resp. the summable) norm. To this end we need to find a biorthogonal system of functionals such that their supports are disjoint with those of the orthogonal vectors, thus guaranteeing the desired one-sided estimates. This is the main technical step in the proof. The estimates going in the opposite direction are then satisfied automatically thanks to the extremal properties of the canonical norms on X.

Proof of Theorem 3. Let $Z \subset X$ be a closed subspace of density character Λ.

By Lemma 6 (for $c_0(\Gamma)$), resp. a result of Rosenthal [14] (for $\ell_1(\Gamma)$) there exist a subspace $Y_1 \subset Z$ which is isomorphic to $c_0(\Lambda)$, resp. $\ell_1(\Lambda)$. So we may assume without loss of generality that $Z = X$.

We start with the case $X = c_0(\Gamma)$. Let Γ be an uncountable cardinal, and let $|||\cdot|||$ be an equivalent norm on $c_0(\Gamma)$. For $\Lambda \subset \Gamma$, denote

$$S_\Lambda = \sup\{|||x||| : x \in c_0(\Gamma), \operatorname{supp}(x) \subset \Lambda, \|x\| \le 1\}.$$

Let $S = \inf_{\Lambda\subset\Gamma,|\Lambda|=|\Gamma|} S_\Lambda$, $\varepsilon > 0$. Choose $\Lambda \subset \Gamma$ such that $S_\Lambda < S + \varepsilon$. For simplicity of notation, we may assume without loss of generality that $\Lambda = \Gamma$. This means, in particular, that $S \le S_\Lambda < S+\varepsilon$, for every $\Lambda \subset \Gamma$. By a simple transfinite induction, choose a transfinite sequence $\{u_\alpha\}_{\alpha=1}^\Gamma$ of disjointly and finitely supported vectors from $c_0(\Gamma)$, $\|u_\alpha\| \le 1$, such that $S \le |||u_\alpha||| < S + \varepsilon$. Indeed, once the initial segment $\{u_\alpha : 1 \le \alpha < \Delta\}$ has been constructed for some $\Delta < \Gamma$, set $\Lambda = \Gamma \setminus \bigcup_{1\le\alpha<\Delta} \operatorname{supp}(u_\alpha)$, and use the property $S \le S_\Lambda < S + \varepsilon$ to find u_Δ.

Choose a sequence $\{f_\alpha\}_{\alpha=1}^\Gamma \subset B_{(\ell_1,|||\cdot|||^*)}$ finitely supported and such that

$$f_\alpha(u_\alpha) > S - \varepsilon.$$

For the rest of the proof we distinguish two cases.

Case 1. Suppose that $\operatorname{cof}(\Gamma) > \omega_0$ (i.e., the cofinality of Γ is an uncountable cardinal). Then, by passing to a suitable subset and reindexing, we may assume in addition that $|\operatorname{supp}(f_\alpha)| = n$ for some fixed $n \in \mathbb{N}$. So we have

$$|\{u_\alpha : u_\alpha \ne u_\beta,\ \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \ne \emptyset\}| \le n, \ \text{ whenever } \ \beta < \Gamma.$$

Our next objective is to pass to a biorthogonal system $\{(u_\alpha, f_\alpha)\}_{\alpha\in\Lambda}$ indexed by a set $\Lambda \subset \Gamma$ of cardinality Γ.

To this end, we first partition the set $\{u_\alpha\}_{\alpha=1}^\Gamma$ using transfinite induction as follows. Let

$$U_1 = \{u_1\} \cup \{u_\alpha : \operatorname{supp}(f_\alpha) \cap \operatorname{supp}(u_1) \ne \emptyset\}.$$

Having found the sets $U_\alpha, \alpha < \Omega < \Gamma$, we let $U_\Omega = \emptyset$ provided $u_\Omega \in \bigcup_{\gamma<\Omega} U_\gamma$, and otherwise we let

$$U_\Omega = \{u_\Omega\} \cup \{u_\alpha : \alpha \in \Gamma \setminus \{\beta : u_\beta \in \bigcup_{\gamma<\Omega} U_\gamma\},\ \operatorname{supp}(f_\alpha) \cap \operatorname{supp}(u_\Omega) \ne \emptyset\}.$$

If the set $\Xi = \{\beta : |U_\beta| = 1\}$ has cardinality $|\Gamma|$, then it is clear that $\operatorname{supp}(u_\alpha) \cap \operatorname{supp}(f_\beta) = \emptyset$ for every distinct $\alpha, \beta \in \Xi$. In this case we are done choosing $\Lambda = \Xi$. Otherwise, we discard the elements $u_\alpha, \alpha \in \Xi$ from future consideration by assuming for simplicity of notation that $\Xi = \emptyset$. Consider now the set

$$W_1 = \bigcup_{\alpha=1}^{\Gamma} (U_\alpha \setminus \{u_\alpha\}).$$

It is clear that $W_1 \subset \{u_\alpha\}_{\alpha=1}^\Gamma$ has cardinality $|\Gamma|$. Moreover,

$$|\{u_\alpha \in W_1 \setminus \{u_\beta\} : \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \ne \emptyset\}| \le n - 1, \ \text{ whenever } \ u_\beta \in W_1.$$

Repeating the previous argument inductively at most n-times, we arrive at the finite sequence of sets $W_n \subset W_{n-1} \subset \cdots \subset W_1$ so that

$$|\{u_\alpha \in W_n \setminus \{u_\beta\} : \operatorname{supp}(f_\beta) \cap \operatorname{supp}(u_\alpha) \neq \emptyset\}| = 0, \quad \text{whenever } \; u_\beta \in W_n.$$

We have found the biorthogonal system by letting $\Lambda = W_n$.

Case 2. Suppose that $\operatorname{cof}(\Gamma) = \omega_0$. We partition the set $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ where $\Gamma_n \nearrow \Gamma$ is an increasing sequence of uncountable regular cardinals [8, pp. 27, 40]. We reindex the original sequence $\{u_\alpha\}_{\alpha=1}^\Gamma$ as a collection $\{u_\alpha^n\}_{\alpha\in\Gamma_n}$, $n \in \mathbb{N}$. Since Γ_n are uncountable and regular, their cofinality $\operatorname{cof}(\Gamma_n)$ is larger than ω_0. By the previous Case 1 we may assume without loss of generality that $|\operatorname{supp}(f_\alpha^n)| = k_n$ is constant for $\alpha \in \Gamma_n$, and $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^n) = \emptyset$ for distinct $\alpha, \beta \in \Gamma_n$. Clearly, by removing a suitable subset of cardinality at most Γ_{m-1} from $\{u_\alpha^m\}_{\alpha\in\Gamma_m}$ we may also assume $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for $\alpha \in \Gamma_n$, $\beta \in \Gamma_m$, $m > n$. It remains to deal with the case $m < n$. We will proceed by induction in n, with replacing the original index sets $\Gamma_1, \dots, \Gamma_n$ with suitable subsets of the same cardinality and such that the condition $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$ will be achieved.

Let us describe the general inductive step for n. We distinguish the following cases. Either there is $m < n$ and a some $u_\beta^m, \beta \in \Gamma_m$ such that the set

$$Q_\beta = \{\alpha \in \Gamma_n : \operatorname{supp}(u_\beta^m) \cap \operatorname{supp}(f_\alpha^n) \neq \emptyset\}$$

has cardinality Γ_n. In this case, remove β from Γ_m, and replace Γ_n by the set Q_β. For n still fixed, this can be repeated at most k_n-times and results in the relation $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$. Note that in this case we have removed at most k_n elements from the original set $\bigcup_{m<n}\{u_\beta^m\}_{\beta\in\Gamma_m}$, and the reduced set Γ_n has the same cardinality as the original one.

Alternatively, during one of the previous finitely many inductive steps, all Q_β have cardinality less than Γ_n. Then we replace Γ_n by $\Gamma_n \setminus \bigcup_{\beta\in\bigcup_{i=1}^{n-1}\Gamma_i} Q_\beta$, which is a set of cardinality Γ_n and leads again to the relation $\operatorname{supp}(f_\alpha^n) \cap \operatorname{supp}(u_\beta^m) = \emptyset$ for all $\alpha \in \Gamma_n, \beta \in \Gamma_m$, $m < n$. Clearly, an inductive step n affects the index sets $\Gamma_m, m < n$ by removing at most finitely many terms, and the cardinality of the reduced Γ_n remains the same. Hence, upon completing the whole induction in n, the final sets Γ_m will have the same cardinality as the original ones. This ends the argument in the case $\operatorname{cof}(\Gamma) = \omega_0$.

Once our system is biorthogonal the result for $c_0(\Gamma)$ follows easily. Indeed, whenever $a_i \in \mathbb{R}, \alpha_i \in \Lambda, i = 1, \dots, k$,

$$\max_{j\in\{1,\dots,k\}} f_{\alpha_j}\Big(\sum_{i=1}^k a_i u_{\alpha_i}\Big) \le (S-\varepsilon)\max_i |a_i| \le |||\sum_{i=1}^k a_i u_{\alpha_i}||| \le (S+\varepsilon) \max_{i\in\{1,\dots,k\}} |a_i|.$$

The argument for $X = \ell_1(\Gamma)$ is easier. Let Γ be an uncountable cardinal, and let $|||\cdot|||$ be an equivalent norm on $\ell_1(\Gamma)$. For $\Lambda \subset \Gamma$ denote

$$S_\Lambda = \inf\{|||x||| : x \in \ell_1(\Gamma), \operatorname{supp}(x) \subset \Lambda, \|x\| \le 1\}.$$

Let $S = \sup_{\Lambda\subset\Gamma,|\Lambda|=|\Gamma|} S_\Lambda$. Choose $\Lambda \subset \Gamma, |\Lambda| = |\Gamma|$ such that $S_\Lambda > S - \frac{\varepsilon}{4}$. For simplicity of notation, we may assume without loos of generality that $\Lambda = \Gamma$. This means, in particular, that $S \ge S_\Lambda \ge S - \frac{\varepsilon}{4}$, for every $\Lambda \subset \Gamma$. By a simple transfinite induction, choose a transfinite sequence $\{u_\alpha\}_{\alpha=1}^\Gamma$ of disjointly and finitely supported vectors from $\ell_1(\Gamma)$, $\|u_\alpha\| \le 1$, such that $S \ge |||u_\alpha||| \ge S - \frac{\varepsilon}{4}$. Indeed, once the initial segment $\{u_\alpha : 1 \leq \alpha < \Delta\}$ has been constructed for some $\Delta < \Gamma$, set $\Lambda = \Gamma \setminus \bigcup_{1\leq\alpha<\Delta} \operatorname{supp}(u_\alpha)$, and use the property $S \geq S_\Lambda \geq S - \frac{\varepsilon}{4}$ to find u_Δ. It is now easy to verify the property

$$S \geq ||| \sum a_i u_{\alpha_i} ||| \geq S - \frac{\varepsilon}{4}$$

whenever $\sum |a_i| = 1$. □

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Bulletin of the Belgian Mathematical Society
Simon Stevin

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Brussels, August 22, 2019.

Dear Professor Novotný

It is our pleasure to inform you that your paper entitled
Some Remarks on Schauder Bases in Lipschitz Free Spaces,
has been accepted for publication in the Bulletin of the Belgian Mathematical Society – Simon Stevin.

Your paper is tentatively scheduled to appear in Vol. 27 issue 2, 2020. You will receive proofs by electronic mail in due time.

With best regards,

Yours Sincerely,

Stefaan Caenepeel,
Editor-in-Chief.

SOME REMARKS ON SCHAUDER BASES IN LIPSCHITZ FREE SPACES

MATĚJ NOVOTNÝ

Abstract. We show that the basis constant of every retractional Schauder basis on the Free space of a graph circle increases with the radius. As a consequence, there exists a uniformly discrete subset $M \subseteq \mathbb{R}^2$ such that $\mathcal{F}(M)$ does not have a retractional Schauder basis. Furthermore, we show that for any net $N \subseteq \mathbb{R}^n$, $n \geq 2$, there is no retractional unconditional basis on the Free space $\mathcal{F}(N)$.

1. Introduction

Let (M, d) be a metric space with a distinguished point $0 \in M$. Denote $\mathrm{Lip}_0(M)$ the space of all Lipschitz functions $f : M \to \mathbb{R}$ with the property $f(0) = 0$. Such a space can be equipped with the Lipschitz norm $\|f\| = \sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)}$, which turns it into a Banach space. We see that each point in M can be naturally embedded into $\mathrm{Lip}_0(M)^*$ via the Dirac mapping δ: $\delta_x(f) = f(x)$, $f \in \mathrm{Lip}_0(M)$, $x \in M$. The norm-closure of the subspace generated by functionals δ_x, $x \in M$, i.e.

$$\overline{\mathrm{span}}^{\mathrm{Lip}_0(M)^*} \{\delta_x | x \in M\}$$

is the Lipschitz Free space over M, denoted $\mathcal{F}(M)$. Lipschitz Free spaces were introduced already by Arens and Eells in [1], although the authors did not use the name Lipschitz Free spaces. Free spaces are called Arens-Eells spaces in [2], where a lot of results regarding the topic is presented.

Lipschitz Free spaces gained a lot of interest in last decades, connecting nonlinear theory with the linear one. Given two pointed metric spaces M, N, every Lipschitz mapping $\varphi : M \to N$ which fixes the point 0 extends to a bounded linear map $F : \mathcal{F}(M) \to \mathcal{F}(N)$, making the following diagram commute:

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{F} & \mathcal{F}(N) \\ {\scriptstyle \delta_M}\big\uparrow & & \big\uparrow{\scriptstyle \delta_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

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We focus on structural properties of Lipschitz Free spaces. It is well-known that $\mathrm{Lip}_0(\mathbb{R}) = L_\infty$, which yields $\mathcal{F}(\mathbb{R}) = L_1$ isometrically and similarly $\mathcal{F}(\mathbb{N}) = \ell_1$. In [3], the authors prove that $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(\mathbb{N})$ if M is infinite (has at least cardinality $\aleph_0$), which was further extended from $\mathbb{N}$ to all cardinalities in [4]. However, $\mathcal{F}(\mathbb{R}^2)$ cannot be embedded in $\mathcal{F}(\mathbb{R}) = L_1$ (see [5]).

Certain results were obtained concerning approximation properties in Free spaces, including [6],[7],[8],[9],[10],[11] and of course [12]. However, not much is known yet about Schauder bases in Free spaces. Hájek and Pernecká [13] constructed a Schauder basis for the Free spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\mathbb{R}^n)$. From [14] we have $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(\mathbb{R}^n)$ for every M with non-empty interior, which gives existence of Schauder basis on such $\mathcal{F}(M)$.

This article follows up the article [4], where the authors proved existence (and in the case of c_0 constructively) of a Schauder basis on $\mathcal{F}(N)$, for any net N in spaces $C(K)$ for K metrizable compact (hence for c_0 and $\mathbb{R}^n$). In section 3 we show that the same construction as in [4] cannot be used for constructing bases in $\mathcal{F}(N)$ for arbitrary uniformly discrete subset N. In section 4 we prove that bases constructed in [4] are not unconditional and that for nets in $\mathbb{R}^n$, no Schauder basis on $\mathcal{F}(N)$ arising from the technique using retractions can be unconditional.

2. Preliminaries

As we mentioned, we are interested in constructing a Schauder basis on Lipschitz Free space. However, constructing such basis directly on the Free space is rather complicated, wherefore we prefer to work with its adjoint space and transfer the results to the Free space. The next theorem shows a way to construct a Schauder basis through operators on $\mathrm{Lip}_0(M)$.

Theorem 1. *Let M be a pointed metric space. Suppose there exists a sequence of linear operators $E_n : \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(M)$, which satisfies the following conditions:*

(1) $\dim E_n (\mathrm{Lip}_0(M)) = n$ *for every* $n \in \mathbb{N}$,

(2) There exists $K > 0$ such that E_n is K-bounded for every $n \in \mathbb{N}$,

(3) $E_m E_n = E_n E_m = E_n$ *for every* $m, n \in \mathbb{N}$, $n \leq m$,

(4) For every n, the operator E_n is continuous with respect to topology of pointwise convergence on $\mathrm{Lip}_0(M)$,

(5) For every $f \in \mathrm{Lip}_0(M)$ the function sequence $E_n f$ converges pointwise to f.

Then the space $\mathcal{F}(M)$ has a Schauder basis with the basis constant at most K.

Proof. Note first that the topology of pointwise convergence coincides with the w^*-topology on bounded subsets of $\mathrm{Lip}_0(M)$. Therefore, from the condition (4), the operators E_n are w^* to w^* continuous on bounded subsets of $\mathrm{Lip}_0(M)$ and hence there exist linear operators $P_n : \mathcal{F}(M) \to \mathcal{F}(M)$ such that $P_n^* = E_n$ for every $n \in \mathbb{N}$. It is now clear that $\|P_n\| \leq K$, $\dim P_n (\mathcal{F}(M)) = n$ and that $P_m P_n = P_n P_m = P_n$ for every $m, n \in \mathbb{N}$, $n \leq m$. Furthermore (5) together with the fact that the topology of pointwise convergence coincides with the w^*-topology on bounded subsets of $\mathrm{Lip}_0(M)$ means, that for every $f \in \mathrm{Lip}_0(M)$ the sequence $E_n f$ converges w^* to f, and that for every $\mu \in \mathcal{F}(M)$ the sequence $P_n \mu$ converges weakly to μ. But that means $\|P_n\mu - \mu\| \to 0$ for every $\mu \in \mathcal{F}(M)$. Indeed, if there were $\mu \in \mathcal{F}(M)$, $c > 0$ and a subsequence P_{n_k}, such that $\|P_{n_k}\mu - \mu\| > c$ for all $k \in \mathbb{N}$, then for every $n \geq n_1$, there exists a $k \in \mathbb{N}$ such that $n \leq n_k$, which yields

$$c < \|P_{n_k}\mu - \mu\| \leq \|P_{n_k}\mu - P_n\mu\| + \|P_n\mu - \mu\| \leq (K+1)\|P_n\mu - \mu\|.$$

From $P_1(\mathcal{F}(M)) \subseteq P_2(\mathcal{F}(M)) \subseteq P_3(\mathcal{F}(M)) \subseteq ...$ we get $E = \bigcup_{n=1}^{\infty} P_n(\mathcal{F}(M))$ is a convex set and as all P_n are commuting projections, we have that $\mu \notin \overline{E}$. Indeed, if $\mu \in \overline{E}$, then there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq E$, such that $x_k \to \mu$. If we choose an increasing sequence of numbers $l_k \in \mathbb{N}$, $l_k > n_1$, which satisfy $P_{l_k}x_k = x_k$, we get that

$$\|P_{l_k}x_k - \mu\| \geq \|P_{l_k}\mu - \mu\| - \|P_{l_k}\mu - P_{l_k}x_k\| \geq \frac{c}{K+1} - K\|\mu - x_k\|.$$

Limiting $k \to \infty$ yields $0 \geq \frac{c}{K+1}$, which is a contradiction. Therefore $\mu \notin \overline{E}$. Hence Hahn-Banach theorem gives us the existence of a linear functional $f \in \mathrm{Lip}_0(M)$, $\|f\| = 1$ with $f|_E = 0$ and $f(\mu) > 0$. But that is a contradiction as $P_n\mu \overset{w}{\to} \mu$. Therefore $P_n\mu \to \mu$.

□

The following corollary appears already in [4], p.12. It gives us a way to construct the Schauder basis on $\mathcal{F}(M)$ only by using the metric space M.

Corollary 2. *Let M be a metric space with a distinguished point 0. Suppose there exists a sequence of distinct points $\{\mu_n\}_{n=0}^{\infty} \subseteq M$, $\mu_0 = 0$, together with a sequence of retractions $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi_n : M \to M$, $n \in \mathbb{N}_0$ which satisfy the following conditions:*
- *(i) $\varphi_n(M) = \{\mu_j\}_{j=0}^{n}$ for every $n \in \mathbb{N}_0$,*
- *(ii) $\overline{\bigcup_{j=0}^{\infty} \{\mu_j\}} = M$,*
- *(iii) There exists $K > 0$ such that φ_n is K-Lipschitz for every $n \in \mathbb{N}_0$,*
- *(iv) $\varphi_m\varphi_n = \varphi_n\varphi_m = \varphi_n$ for every $m, n \in \mathbb{N}_0$, $n \leq m$.*

Then the space $\mathcal{F}(M)$ has a Schauder basis with the basis constant at most K.

Proof. It is not difficult to see that for each $n \in \mathbb{N}$ the formula $E_n f = f \circ \varphi_n$, $f \in \mathrm{Lip}_0(M)$ defines a linear operator $E_n : \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(M)$, such that the sequence E_n satisfies the assumptions of Theorem 1.

□

The last two theorems lead us to the following definition.

Definition 1. *Let M be an infinite metric space such that $\mathcal{F}(M)$ has a Schauder basis E with projections P_n, $n \in \mathbb{N}$. We say E is an extensional Schauder basis if there exist finite sets $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq ...$ such that $\bigcup_{n=1}^{\infty} M_n$ is dense in M and we have that for every $n \in \mathbb{N}$ the adjoint P_n^* is a linear extension operator $P_n^* : \mathrm{Lip}_0(M_n) \to \mathrm{Lip}_0(M)$ with $P_n^* f|_{M_n} = f$ (or equivalently P_n is a projection onto $\mathcal{F}(M_n)$). We say E is a retractional Schauder basis, if there exist retractions $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi_n : M \to M$ which satisfy the conditions of Corollary 2 and such that they give rise to the basis E, i.e. the adjoints P_n^* satisfy $P_n^* f = f \circ \varphi_n$, $f \in \mathrm{Lip}_0(M)$.*

It is clear that in the definition we actually have $|M_n \setminus M_{n-1}| = 1$ for every $n \in \mathbb{N}$. Note also that every retractional Schauder basis is a special case of an extensional Schauder basis. The next lemma shows in more detail what form the basis vectors take.

Lemma 3. *Let M be a metric space such that there is a sequence of distinct points $0 = \mu_0, \mu_1, \mu_2, ... \in M$ such that $\bigcup_{n=1}^{\infty} \{\mu_0, \mu_1, ..., \mu_n\}$ is dense in M. For every $n \in \mathbb{N}_0$ denote $M_n = \{\mu_0, ..., \mu_n\}$. Suppose $\mathcal{F}(M)$ has a Schauder basis $B = \{e_n\}_{n=1}^{\infty}$. Then the following are equivalent:*

(1) B is an extensional Schauder basis with extension operators $E_n : \mathrm{Lip}_0(M_n) \to \mathrm{Lip}_0(M)$.
(2) For every $n \in \mathbb{N}$, there are constants $0 \neq c_n, a_i^n \in \mathbb{R}$, $i \in \{1, ..., n-1\}$ such that we have $c_n e_n = \delta_{\mu_n} - \sum_{i=1}^{n-1} a_i^n \delta_{\mu_i}$.

Proof. (2) ⇒ (1). Note first that for every $n \in \mathbb{N}$, we have $e_n \in \mathrm{Im}\, P_n \cap \ker P_{n-1}$. From that it follows inductively for every $n \in \mathbb{N}$ that $\mathrm{Im}\, P_n = \mathrm{span}\,\{\delta_{\mu_1}, ..., \delta_{\mu_n}\}$.

(1) ⇒ (2) The fact that $E_n = P_n^*$ is a bounded linear extension from M_n to M implies that each P_n maps $\mathcal{F}(M)$ onto $\mathcal{F}(M_n)$, which means each basis vector e_n is a linear combination of Dirac functionals at the points of M_n, such that the coefficients at δ_{μ_n} do not vanish. □

Keeping the notation from previous lemma, we see that for each $n \in \mathbb{N}$ we may define a finite dimensional operator $R_n : \mathrm{Lip}_0(M_{n-1}) \to \mathrm{Lip}_0(M_n)$ via

$$R_n f(\mu_j) = \begin{cases} f(\mu_j) & j \in \{0, ..., n-1\} \, , \\ \sum_{i=1}^{n-1} a_i^n f(\mu_i) & j = n \, . \end{cases}$$

The operator $E_n = P_n^*$ can be then reconstructed through a w^*-limit of operator composition $\lim_k R_k R_{k-1} ... R_{n+1}$. The constants c_n were in the lemma only for scaling of the basis vectors e_n.

In case of a retractional basis, the basis vectors take form of two-point molecules: For every $n \in \mathbb{N}$ and $i \in \{1, ..., n-1\}$ exactly one of the coefficients a_i^n is non-zero, namely has the value 1. If for example $a_j^n = 1$, then $\varphi_j(\mu_n) = \mu_j$, which means $e_n = \delta_{\mu_n} - \delta_{\mu_j}$.

Throughout this article, given a metric space M, d will denote its metric. If M is a countable (even finite) uniformly discrete metric space with $\mathcal{F}(M)$ having a retractional Schauder basis, by symbols $\mu_0, \mu_1, \mu_2 ...$, resp. $\varphi_0, \varphi_1, \varphi_2, ...$ we will always mean points $\mu_i \in M$, resp. retractions $\varphi_i : M \to M$ which satisfy Corollary 2. Obviously the finite analogues of Corollary 2 and Theorem 1 also hold. We are going to look in more detail on some properties of retractional Schauder basis.

Following the notation of Corollary 2 (or the proof of Lemma 14 in [4]) we find useful to denote the set-valued functions $F_i = \varphi_i^{-1} : M \to 2^M$, $F_i(x) = \{y | \; \varphi_i(y) = x\}$, $i \in \mathbb{N}_0$. Clearly $F_0(0) = M$. From the commutativity of the φ_i's further follows that for any $i < j$ we have

$$F_i(\mu_i) \cap F_j(\mu_j) \in \{\emptyset, F_j(\mu_j)\} \, .$$

Definition 2. *Let M and μ_i, φ_i, $i \in \mathbb{N}_0$ satisfy the assumptions from Corollary 2 and $M = \{\mu_i\}_{i=0}^{\infty}$. A finite or infinite sequence of points $(\mu_{k_1}, \mu_{k_2}, \mu_{k_3}, ...)$ is called a chain whenever $k_1 < k_2 < k_3 < ...$ and $\varphi_{k_i-1}(\mu_{k_i}) = \mu_{k_{i-1}}$ for every $i \in \{2, 3, ...\}$.*

Note that for every chain $(\mu_{k_1}, \mu_{k_2}, \mu_{k_3}...)$ we have $F_{k_1}(\mu_{k_1}) \supseteq F_{k_2}(\mu_{k_2}) \supseteq F_{k_3}(\mu_{k_3}) \supseteq ....$ We can also introduce partial order on M by $\mu_i \prec \mu_j$ if and only if there exist $n \in \mathbb{N}_0$ points $\mu_{k_1}, ..., \mu_{k_n} \in M$ such that $(\mu_i, \mu_{k_1}, ..., \mu_{k_n}, \mu_j)$ is a chain. Note also that for two chains S, T the difference $S \setminus T$ and intersection $S \cap T$ are also chains, if nonempty. For a finite chain $(x_1, x_2, ..., x_n)$ we call the point x_1 its initial point and x_n its final point.

Every chain can be viewed as a path or its segment from $0 \in M$ to a given point $x \in M$. Indeed, for every $x \in M$ there exists $n \in \mathbb{N}$ such that for every $i \geq n$ one has $\varphi_i(x) = x$. Assuming n is the least number with that property we can define the set $T_0^x = \bigcup_{i=0}^n \{\varphi_i(x)\}$ which contains exactly the points of the chain with initial point 0 and final point x. Regarding T_0^x as an ordered set (the order $\prec$ is linear on T_0^x), it is clear that given $x \in M$, there exists exactly one chain T_0^x from 0 to x.

Note also that for every chain $(\mu_{n_1}, ..., \mu_{n_k})$, $k \geq 2$ there exist constants c_{n_i} (the constants from Lemma 3) such that for basis vectors $e_{n_1}, ..., e_{n_k}$ we have

$$\sum_{i=2}^{k} c_{n_i} e_{n_i} = \delta_{\mu_{n_k}} - \delta_{\mu_{n_1}}.$$

The following lemma says that, if the space is not too "porous", basis vectors can be made only of two-point molecules in points which are not too far from each other.

Lemma 4 (Step lemma)**.** *Let M be a countable metric space, $\alpha > 0, K \geq 1$ and $\varphi_n : M \to M$ a system of retractions from Corollary 2. If $(\mu_{i_1}, ..., \mu_{i_j})$, $j > 1$ is a chain and there exist distinct points $x_1, ..., x_k \in M$ with $d(x_l, x_{l+1}) \leq \alpha$, $l \in \{1, ..., k-1\}$, $x_1 = \mu_{i_j}$, $x_k = \mu_{i_1}$ and $\sup_{i_1 \leq n \leq i_j} \operatorname{Lip} \varphi_n \leq K$, then $d(\mu_{i_{m-1}}, \mu_{i_m}) \leq 2K\alpha$ for all $m \in \{2, ..., j\}$.*

Proof. Suppose $d(\mu_{i_{m-1}}, \mu_{i_m}) > 2K\alpha$ for some $m \in \{2, ..., j\}$. We know $\varphi_{i_m}(\mu_{i_j}) = \mu_{i_m}$ and $\varphi_{i_m-1}(\mu_{i_j}) = \mu_{i_{m-1}}$. We prove by induction for all $l \in \{1, ..., k\}$ that $\varphi_{i_m}(x_l) = \mu_{i_m}$ and $\varphi_{i_m-1}(x_l) = \mu_{i_{m-1}}$, which is a contradiction as $x_k = \mu_{i_1}$ and $\varphi_{i_m}(\mu_{i_1}) = \mu_{i_1} \neq \mu_{i_m}$. For $l = 1$ we have $x_l = \mu_{i_j}$ and the statement clearly holds. Suppose it holds for all $l = 1, ..., s-1 < k$. From $d(x_{s-1}, x_s) \leq \alpha$ it follows that $d(\mu_{i_m}, \varphi_{i_m}(x_s)) \leq K\alpha$ and $d(\mu_{i_{m-1}}, \varphi_{i_m-1}(x_s)) \leq K\alpha$. From commutativity of all φ_n's follows that either $\varphi_{i_m-1}(x_s) = \varphi_{i_m}(x_s) \notin \{\mu_{i_m}\}$ holds or $\varphi_{i_m}(x_s) = \mu_{i_m}$ and $\varphi_{i_m-1}(x_s) = \mu_{i_{m-1}}$ is true. Since $B_{K\alpha}(\mu_{i_m}) \cap B_{K\alpha}(\mu_{i_{m-1}}) = \emptyset$ we conclude the latter is true, which completes the induction step and the contradiction is obtained. □

In the following section, we are going to prove that there are spaces $\mathcal{F}(M)$ with no retractional Schauder basis yet having Schauder basis, moreover extensional.

3. Nonexistence of retractional Schauder bases

Definition 3. *Let $x_0, x_1, ..., x_n$, $n \in \mathbb{N}$ be distinct points. The set $C_n^0 = \{x_0, x_1, x_2..., x_n\}$ with the (standard graph) metric $d(x_k, x_0) = n$, $k \neq 0$, $d(x_k, x_l) = \min\{|k-l|, n - |k-l|\}$, $k, l > 0$ we call a circle or a circle of radius n with centre x_0.*

In the following, we regard the centre x_0 as the base point in the pointed metric space (C_n^0, d, x_0) and denote it 0.

We are also going to use an uncentered circle, i.e. a subgraph $C_n = \{x_1, x_2, ..., x_n\}$ with the induced metric. On C_n, we define orientation: We say point x_l lies to the left of the point x_k, $k, l \in \{1, ..., n\}$, if one of these situations happens:

(1) $k > \frac{n-1}{2}$ and $l \in \left\{k, k-1, ..., k - \lfloor \frac{n+1}{2} \rfloor + 1\right\}$,

(2) $k \leq \frac{n-1}{2}$ and $l \in \{k, k-1, ..., 1\} \cup \left\{n, n-1, ..., n - \lfloor \frac{n+1}{2} \rfloor + k + 1\right\}$.

Analogously, we say x_l lies to the right of x_k if one of the following conditions is satisfied:

(1) $k \leq \frac{n-1}{2}$ and $l \in \left\{k, k+1, ..., k + \lfloor \frac{n+1}{2} \rfloor\right\}$,

(2) $k > \frac{n-1}{2}$ and $l \in \{k, k+1, ..., n\} \cup \left\{1, 2, ..., \lfloor \frac{n+1}{2} \rfloor - (n-k+1)\right\}$.

We show that every retractional Schauder basis on $\mathcal{F}(C_n^0)$ has a basis constant which is increasing with n.

Theorem 5. *Let $n \in \mathbb{N}$, $n \geq 10$ and let $\{\varphi_i\}_{i=0}^n$ be a system of retractions on a circle C_n^0 satisfying the conditions of Corollary 2 . Then there is an $s \in \{1, ..., n\}$ such that*

$$\operatorname{Lip} \varphi_s \geq \frac{\sqrt{8n+1} - 1}{8}.$$

Proof. Let us fix $n \geq 10$ and denote $K = \frac{\sqrt{8n+1}-1}{8}$. We have $\mu_0 = 0$ and $\mu_1 \in C_n$ with $\varphi_1(x) = \mu_1$ for all $x \in C_n$ and $\varphi_1(0) = 0$. Indeed, if $\varphi_1(y) = 0$ for some $y \in C_n$, then the sets $F_1(0)$ and $F_1(\mu_1)$ have distance 1. Since they are finite, there exist $w \in F_1(0)$, $z \in F_1(\mu_1)$ such that $d(w, z) = 1$ and clearly $d(\varphi_1(w), \varphi_1(z)) = n$, which trivially yields the result, as $n > K$. We prove the theorem by contradiction and assume therefore, $\operatorname{Lip} \varphi_i < K$ for all $i \in \{1, ..., n\}$.

For every point $x \in C_n$ there exists a $k \in \{1, ..., n\}$ such that $\{x\} = \{\mu_k\} = \varphi_k(C_n) \setminus \varphi_{k-1}(C_n)$ and therefore there exists exactly one chain $S_x = (\mu_{k_1}, \mu_{k_2}, ..., \mu_{k_l})$, such that $\mu_{k_1} = \mu_1$ and $\mu_{k_l} = x$ (equivalently $k_1 = 1$ and $k_l = k$).

Let us introduce sets

$$A = \{y|\ y \in C_n \setminus \{\mu_1\}, d(y, \mu_1) \leq 3K, y \text{ lies to the left of } \mu_1\}$$

$$B = \{y|\ y \in C_n \setminus \{\mu_1\}, d(y, \mu_1) \leq 3K, y \text{ lies to the right of } \mu_1\}$$

and a mapping $f : C_n \setminus \{\mu_1\} \to \{A, B\}$,

$$f(w) = \begin{cases} A & \text{there is a } z \in S_w \cap A \text{ such that for every } y \in S_w,\ z \prec y, \text{ we have } y \notin A \cup B\,, \\ B & \text{there is a } z \in S_w \cap B \text{ such that for every } y \in S_w,\ z \prec y, \text{ we have } y \notin A \cup B\,. \end{cases}$$

Note that the definitions of A, B make perfect sense, as $3K < \frac{n}{2}$. Also, the mapping f is well-defined, as for every $w \in C_n \setminus \{\mu_1\}$ the intersection $S_w \cap (A \cup B)$ is nonempty. Indeed, according to Step lemma 4 applied on the C_n, the distance between any two adjacent points in a chain is smaller than $2K$ and therefore for the second element $z \in S_w$ (meaning $S_w = (\mu_1, z, ..., w)$) we have $d(\mu_1, z) \leq 2K$ and thus $z \in A$ or $z \in B$.

Observe that $f(w) = A$ for every $w \in A$ and $f(w) = B$ for every $w \in B$. We prove there exist two points $a, b \in C_n \setminus (\{\mu_1\} \cup A \cup B)$ such that $d(a, b) = 1$, $f(a) = A$ and $f(b) = B$.

Let us assume for contradiction that $f(w) = A$ for all points $w \in C_n \setminus (\{\mu_1\} \cup A \cup B)$. Denote z the closest point to the right of the set B, i.e. the only point with $3K < d(z, \mu_1) \leq 3K + 1$ and $d(z, B) = 1$. We have $f(z) = A$, which means the chain $S_z = (\mu_1, \mu_{k_2}, ..., z)$ leaves the set A and goes to the left around (meaning omitting the set $A \cup B$) the circle to the point z, with steps smaller than $2K$. Therefore there exists a point $\mu_l \in S_z$ such that $d(\mu_1, \mu_l) \geq \frac{n-2K}{2}$. But then we have $\varphi_l(\mu_1) = \mu_1$, $\varphi_l(z) = \mu_l$, which yields

$$\operatorname{Lip} \varphi_l \geq \frac{d(\mu_1, \mu_l)}{d(\mu_1, z)} \geq \frac{n - 2K}{2(3K + 1)} \geq K,$$

as $n \geq 10$, which contradicts our assumption.

Therefore, let there exist two points $a, b \in C_n \setminus (\{\mu_1\} \cup A \cup B)$ such that $d(a, b) = 1$, $f(a) = A$ and $f(b) = B$. Consider the two chains $S_a = (\mu_1, \mu_{k_1}, \mu_{k_2} ..., a)$ and $S_b = (\mu_1, \mu_{l_1}, \mu_{l_2} ..., b)$ and let i and j be such that $\mu_{k_i} \in A$, $\mu_{l_j} \in B$ and we have $\mu, \nu \notin A \cup B \cup \{\mu_1\}$ for every $\mu \in S_a$, $\mu_{k_i} \prec \mu$ and every $\nu \in S_b$, $\mu_{l_j} \prec \nu$. Note that $d(\mu_1, \mu_{l_j}) \geq K + 1$ and $d(\mu_1, \mu_{k_i}) \geq K + 1$.

Without loss of generality suppose $k_i < l_j$. Then $\varphi_{l_j}(b) = \mu_{l_j}$ and $d(\varphi_{l_j}(a), \mu_{l_j}) \leq K$. This implies $u_a := \varphi_{l_j}(a)$ has distance at most K from the set B, which yields $d(\mu_1, u_a) \leq 4K$ and the chain $S = (\mu_{k_i}, ..., u_a)$ must go from the set A to the left around the circle closer to the set B. Thus there must exist a point $v = \mu_s \in S$ such that $d(v, \mu_1) \geq \frac{n-2K}{2}$. It follows that

$$\operatorname{Lip} \varphi_s \geq \frac{d(\varphi_s(u_a), \varphi_s(\mu_1))}{d(u_a, \mu_1)} = \frac{d(v, \mu_1)}{d(u_a, \mu_1)} \geq \frac{n - 2K}{8K} = K,$$

which is again a contradiction. We conclude there exists an $s \in \{1, ..., n\}$ such that $\operatorname{Lip} \varphi_s \geq K = \frac{\sqrt{8n+1}-1}{8}$.

□

Corollary 6. *There exists a uniformly discrete set $N \subseteq \mathbb{R}^2$ such that the Free space $\mathcal{F}(N)$ has no retractional Schauder basis.*

Proof. Let $N = \bigcup_{n=1}^{\infty} C_{4^n}^0$ be a union of circles with the same centre 0 and with radii 4^n, $n \in \mathbb{N}$. Suppose d_n is the metric on $C_{4^n}^0$. Let us define a metric on N in the following way:

$$d(x, y) = \begin{cases} d_n(x, y) & \text{if } x, y \in C_{4^n}^0 \,, \\ \max\{4^i, 4^j\} & \text{if } x \in C_{4^i}, y \in C_{4^j}, i \neq j \,. \end{cases}$$

It is clear that d is indeed a metric on N and one has no difficulties to embed N into $\mathbb{R}^2$ in a bilipschitz way, actually with distortion not worse than 2π. We show that every sequence of retractions $\varphi_i : N \to N$ satisfying conditions (i) and (iv) from Corollary 2 cannot satisfy the condition (iii) of that corollary.

Let therefore $\varphi_i : N \to N$ be a commuting sequence of retractions such that $\varphi_0(0) = 0$ and $|\varphi_i(N)| = i + 1$. We show that for every $k \in \mathbb{N}$, $k \geq 4$, there exists an $n = n_k \in \mathbb{N}$ such that $\operatorname{Lip} \varphi_{n_k} \geq k$. Pick therefore $k \in \mathbb{N}$, $k \geq 4$, and find the smallest n such that $\mu_n \in C_{4^k}$. Then $\varphi_i(\mu_n) = \mu_n$ for every $i \geq n$. If there exist $j \geq n$ and $x \in C_{4^k}$ such that $\varphi_j(x) \notin C_{4^k}$, we have $\operatorname{Lip} \varphi_j \geq 4^k \geq k$ and the proof is finished. Indeed, if we take $x \in C_{4^k}$

such that $\varphi_j(x) \notin C_{4^k}$ and without loss of generality we assume x is such that $d(x, \mu_n)$ is minimal among all $x \in C_{4^k}$ with $\varphi_j(x) \notin C_{4^k}$, we have $d(\varphi_j(y), \varphi_j(x)) \geq 4^k \geq k$ for one of x's neighbours y (i.e. $d(x, y) = 1$). This means $\operatorname{Lip} \varphi_j \geq k$. If, on the contrary, we have $\varphi_i(x) \in C_{4^k}$ for all $i \geq n$ and all $x \in C_{4^k}$, we find ourselves in the case of Theorem 5. Indeed, if we view the circle $C^0_{4^k}$ as a set $C^0_{4^k} = \{0, \mu_{s_1}, \mu_{s_2}, ..., \mu_{s_{4^k}}\}$ (for some eligible $s_1, s_2, ..., s_{4^k} \in \mathbb{N}$) and look only at retractions $\varphi_0, \varphi_{s_1}, \varphi_{s_2}, ..., \varphi_{s_{4^k}}$ restricted to the circle $C^0_{4^k}$, we apply 5 and conclude $\max\{\operatorname{Lip} \varphi_{s_1}, \operatorname{Lip} \varphi_{s_2}, ..., \operatorname{Lip} \varphi_{s_{4^k}}\} \geq \frac{\sqrt{8 \cdot 4^k + 1} - 1}{8} \geq k$. □

We see it is impossible to build a retractional Schauder basis on $\mathcal{F}(N)$. However, the space $\mathcal{F}(N)$ has an extensional Schauder basis as we are going to show in the next proposition:

Proposition 7. *Let $N = \bigcup_{n=1}^{\infty} C^0_{4^n}$ be the metric space from Corollary 6. Then $\mathcal{F}(N)$ has an extensional monotone Schauder basis.*

Proof. First, note that we have orientation of every C_{4^n}, $n \in \mathbb{N}$. For every $i \in \mathbb{N}$, define $k = k(i)$ as the unique integer such that $\frac{4^k - 1}{3} \leq i < \frac{4^{k+1} - 1}{3}$. Let $N = \{0, x_1, x_2, x_3, ...\}$ be enumerated in such way that for every $i \in \mathbb{N}$ we have $x_i \in C_{4^k}$ and that the enumeration respects the orientation on every circle C_{4^k}. Namely, if $x_i, x_{i+1} \in C_{4^k}$, we have that $d(x_i, x_{i+1}) = 1$ and x_{i+1} lies to the right of x_i. Denote $D_i = \{0, x_1, x_2, ...x_i\}$.

We are going to define a sequence of extension operators $P_i : \operatorname{Lip}_0(D_i) \to \operatorname{Lip}_0(N)$ and prove they satisfy the assumptions of Theorem 1. In order to do that, let us define some preparatory notions. Define the left and the right "D_i-neighbour" functions $\nu^l_i, \nu^r_i : \bigcup_{n=1}^{k(i)} C^0_{4^n} \to D_i$ as follows: For each $n \in \{1, 2, ..., k(i)\}$ and $x \in C_{4^n}$, let $\nu^l_i(x) \in D_i$ be the closest point to the left of x and let $\nu^r_i(x) \in D_i$ be the closest point to the right of x. We set $\nu^l_i(0) = \nu^r_i(0) = 0$. Note that $\nu^l_i(x) = \nu^r_i(x) = x$ if and only if $x \in D_i$. Further we need to define "right-" and "left-" metric function (not proper metrics) on every circle C_{4^n}. For points $x, y \in C_{4^n}$ we set the value $d^l(x, y)$ as the length of the path (in the graph C_{4^n}) going from x to the left up to y. Analogously, we set $d^r(x, y)$ as the length of the path going from x to the right up to y. It is clear that for $x, y \in C_{4^n}$ we have $d^l(x, y) = d^r(y, x)$ and $d(x, y) = \min\{d^r(x, y), d^l(x, y)\}$.

Further we define for every $i \in \mathbb{N}$ the i-th interpolation function $I_i : \operatorname{Lip}_0(D_i) \times \bigcup_{n=1}^{k(i)} C^0_{4^n} \to \mathbb{R}$ via

$$I_i(f, x) = \frac{d^r(x, \nu^r_i(x)) f(\nu^l_i(x)) + d^l(x, \nu^l_i(x)) f(\nu^r_i(x))}{d^l(x, \nu^l_i(x)) + d^r(x, \nu^r_i(x))} \quad \text{if } x \neq \nu^l_i(x) \text{ or } x \neq \nu^r_i(x)$$

and $I_i(f, x) = f(x)$ for $x = \nu^l_i(x) = \nu^r_i(x)$. Clearly, $I_i(f, x)$ is the value of linear interpolation of the function f between closest points of x to the left and to the right from the set D_i, given we take x itself to be the closest point to x in any direction if $x \in D_i$. Let now $f \in \operatorname{Lip}_0(D_i)$. Then we define our (extension) operators P_i, $i \in \mathbb{N}$ simply as

$$P_i f(x) = \begin{cases} I_i(f, x) & x \in \bigcup_{n=1}^{k(i)} C^0_{4^n}, \\ 0 & x \in \bigcup_{n=k(i)+1}^{\infty} C_{4^n} \end{cases}$$

and of course, $P_0 = 0$. Clearly P_i is a linear operator for every $i \in \mathbb{N}$ and the function $P_i f$ is Lipschitz with the same constant as f. Indeed, if we take $x \in C_{4^n}$ and $y \in C_{4^m}$ with $m < n$, we see from the definition of I_i that $\min_{z \in C_{4^n}} f(z) \le P_i f(x) \le \max_{z \in C_{4^n}} f(z)$ and $\min_{w \in C_{4^m}} f(w) \le P_i f(y) \le \max_{w \in C_{4^m}} f(w)$. From that and from the fact that $d(z, w) = d(x, y) = 4^n$ holds for all $z \in C_{4^n}$ and $w \in C_{4^m}$, we get

$$|P_i f(x) - P_i f(y)| \le \max_{\substack{z \in C_{4^n} \\ w \in C_{4^m}}} |f(z) - f(w)| \le 4^n \|f\| = d(x, y)\|f\|.$$

For $x \in C_{4^n}$ and 0 we have clearly $|P_i f(x) - P_i f(0)| \le \max_{z \in C_{4^n}} |f(z)| \le d(x, 0)\|f\|$.

The only nontrivial case to prove is the case $x, y \in C_{4^{k(i)}}$. Let therefore $x, y \in C_{4^{k(i)}}$. There are three cases. If $x, y \in D_i$, then $P_i f(x) = f(x)$ and $P_i f(y) = f(y)$, which is trivial. Let $x, y \notin D_i$ and $\nu_i^r(x) = \nu_i^r(y) = a$, $\nu_i^l(x) = \nu_i^l(y) = b$. We can assume $d^r(b, x) \le d^r(b, y)$, for the roles of x and y are symetrical. From that we have $d^r(y, a) \le d^r(x, a)$. If $d(x, y) = d^r(x, y)$, we have

$$\begin{aligned} |P_i f(x) - P_i f(y)| &= \left| \frac{f(b)d^r(x, a) + f(a)d^r(b, x)}{d^r(b, a)} - \frac{f(b)d^r(y, a) + f(a)d^r(b, y)}{d^r(b, a)} \right| \\ &= \left| \frac{f(b)d^r(x, y) - f(a)d^r(x, y)}{d^r(b, a)} \right| \\ &\le \frac{\|f\|d(a, b)}{d^r(b, a)} d^r(x, y) \le \|f\|d(x, y). \end{aligned}$$

If $d(x, y) = d^l(x, y)$, then $d(x, y) = d^r(y, a) + d^r(a, b) + d^r(b, x)$ and then from $d^r(b, a) - d^r(x, y) = d^r(b, x) + d^r(y, a)$ we have by triangle inequality

$$\begin{aligned} |P_i f(x) - P_i f(y)| &= \left| \frac{f(b)d^r(x, y) - f(a)d^r(x, y)}{d^r(b, a)} \right| \\ &= \left| \frac{f(b)\left(d^r(x, y) - d^r(b, a)\right) + (f(b) - f(a))d^r(b, a) + f(a)\left(d^r(b, a) - d^r(x, y)\right)}{d^r(b, a)} \right| \\ &= \left| \frac{\left(d^r(b, x) + d^r(y, a)\right)\left(f(a) - f(b)\right) + (f(b) - f(a))d^r(b, a)}{d^r(b, a)} \right| \\ &\le \|f\| \left(\frac{d(a, b)}{d^r(a, b)} \left(d^r(b, x) + d^r(y, a)\right) + d(a, b) \right) \le \|f\|d(x, y) \end{aligned}$$

The case $x \in D_i$, $y \notin D_i$ is proved in a similar way.

We see that the functions $f_j = P_i\left(\chi_{\{x_j\}}\right)$, $1 \le j \le i$ create a basis of each space $P_i(\mathrm{Lip}_0(N))$, hence $\dim P_i(\mathrm{Lip}_0(N)) = i$ for every $i \in \mathbb{N}$.

To prove the commutativity it suffices to prove $P_{i+1}P_i = P_i P_{i+1} = P_i$ for every $i \in \mathbb{N}$. While $P_i P_{i+1} = P_i$ is clear, we prove for every $f \in \mathrm{Lip}_0(N)$ we have $P_{i+1}P_i f = P_i f$. Fix $f \in \mathrm{Lip}_0(N)$. If $k(i+1) > k(i)$, then $D_i = \bigcup_{n=1}^{k(i)} C_{4^n}^0$ and $P_{i+1}P_i f(x) = f(x) = P_i f(x)$ for all $x \in \bigcup_{n=1}^{k(i)} C_{4^n}^0$ and $P_{i+1}P_i f(x) = 0 = P_i f(x)$ for all $x \notin \bigcup_{n=1}^{k(i)} C_{4^n}^0$. Let therefore $k(i+1) = k(i)$.

Denote $a = x_i = \nu_i^l(x_{i+1})$ and $b = \nu_i^r(x_{i+1})$. All we need to check is $P_{i+1}P_if(y) = P_if(y)$ holds for all $y \in C_{4^{k(i)}} \setminus D_i$. Indeed, for all other points x we have $P_if(x) = P_{i+1}f(x)$. Take therefore a point $y \neq x_{i+1}$ (otherwise it is trivial). Note that $\nu_i^l(y) = a$, $\nu_i^r(y) = b$ and that $d^r(a, x_{i+1}) = 1$. Then we have

$$\begin{aligned}
P_{i+1}(P_if)(y) &= \frac{d^r(x_{i+1}, y)P_if(b) + d^r(y,b)P_if(x_{i+1})}{d^r(x_{i+1}, b)} \\
&= \frac{d^r(x_{i+1}, y)f(b) + d^r(y,b)\cdot \frac{f(b)+d^r(x_{i+1},b)f(a)}{d^r(a,b)}}{d^r(x_{i+1}, b)} \\
&= \frac{d^r(x_{i+1}, y)d^r(a,b) + d^r(y,b)}{d^r(x_{i+1}, b)d^r(a,b)} \cdot f(b) + \frac{d^r(y,b)}{d^r(a,b)} \cdot f(a) \\
&= \frac{d^r(x_{i+1}, y)d^r(x_{i+1},b) + d^r(x_{i+1},y) + d^r(y,b)}{d^r(x_{i+1}, b)d^r(a,b)} \cdot f(b) + \frac{d^r(y,b)}{d^r(a,b)} \cdot f(a) \\
&= \frac{d^r(x_{i+1}, b)\left(1 + d^r(x_{i+1},y)\right)}{d^r(x_{i+1}, b)d^r(a,b)} \cdot f(b) + \frac{d^r(a,y)}{d^r(a,b)} \cdot f(a) \\
&= \frac{d^r(a,y)}{d^r(a,b)} \cdot f(b) + \frac{d^r(a,y)}{d^r(a,b)} \cdot f(a) \\
&= P_if(y)
\end{aligned}$$

and the commutativity is proved.

Let $i \in \mathbb{N}$. If $f_\alpha \to f$ pointwise, then for every $x \in D_i$ we have $P_if_\alpha(x) = f_\alpha(x) \to f(x) = P_if(x)$ and for every $x \in \bigcup_{l=k(i)+1}^{\infty} C_{4^l}$ we have $P_if_\alpha(x) = 0 = P_if(x)$. Finally, for every $x \in C_{4^{k(i)}} \setminus D_i$ we have $P_if_\alpha(x) = \gamma_x f_\alpha(a_x) + (1-\gamma_x)f_\alpha(b_x)$, for some eligible $\gamma_x \in [0,1]$, $a_x, b_x \in D_i$ and the choice of these points depends only on x (and i of course). Therefore $P_if_\alpha \to P_if$ pointwise, which means that every operator P_i is continuous with respect to topology of pointwise convergence.

Finally the sequence P_if converges pointwise to f. Indeed, for every $y \in N$ there exists $i \in \mathbb{N}$ such that $y \in D_i \subseteq D_{i+1} \subseteq D_{i+2} \dots$, which yields $P_if(y) = P_jf(y) = f(y)$ for all $j \geq i$. Hence $P_if \to f$ pointwise.

Since the operators P_i meet all assumptions from Theorem 1, we get that there is a sequence of operators $T_i : \mathcal{F}(N) \to \mathcal{F}(N)$, $i \in \mathbb{N}_0$ with $T_i^* = P_i$ which build a monotone Schauder basis for $\mathcal{F}(N)$.

□

Remark. It was not necessary for the construction of P_i's to enumerate the set N with respect to orientation on every circle C_{4^k}. Actually any enumeration which satisfies $x_i \in C_{4^k}$ for every $i \in \mathbb{N}$ works. Our choice only slightly simplifies the proof.

4. Unconditionality of retractional Schauder bases

As we construct a Schauder basis on $\mathcal{F}(M)$ via sequence of retractions, as described in Corollary 2, properties of such a basis depend also on properties of the metric space M. Naturally it leads us to the question: What can M be like such that there is an unconditional retractional Schauder basis on $\mathcal{F}(M)$? The next lemma sets a condition on the chains under which the acquired basis is conditional. It is further used in Theorem 10, which shows that retractional bases on Free spaces of nets in finite-dimensional spaces are conditional.

Lemma 8. *Let $\alpha, \beta > 0$ and let N be an α-separated metric space, such that there exist retractions $\varphi_i : N \to N$ satisfying the conditions from Corollary 2. Suppose there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$ there exist chains $S = (\mu_0, \mu_{k_1}, ..., \mu_{k_s})$ and $T = (\mu_0, \mu_{l_1}, ..., \mu_{l_m})$, $s, m \in \mathbb{N}$ with $d(\mu_{k_s}, \mu_{l_m}) \leq \beta$ and $|S \setminus T| \geq n$. Then the retractional Schauder basis on $\mathcal{F}(N)$ corresponding to the retractions φ_i is conditional.*

Proof. Let now P_i be the associated Schauder projection to the mapping φ_i for each $i \in \mathbb{N}_0$, i.e. the projection to the subspace span $\{\delta_{\mu_0}, \delta_{\mu_1}, ..., \delta_{\mu_i}\}$. Instead of working directly with $P_0, P_1, P_2, ...$ we will use their adjoints $P_0^*, P_1^*, P_2^*, ...$ and for every $n \in \mathbb{N}$, $n \geq n_0$ we construct a function $f_n \in \mathrm{Lip}_0(N)$ with $\|f_n\| \leq 1$ and find a sequence of signs $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{k_s}$ for some $s \geq n$ such that the following inequality holds

$$\left\| \sum_{i=0}^{k_s} \varepsilon_i (P_{i+1}^* - P_i^*) f_n \right\| \geq \frac{\alpha(n-1)}{\beta}.$$

Fix $n \in \mathbb{N}$ and chains $S = (\mu_0, \mu_{k_1}, ..., \mu_{k_s})$, $T = (\mu_0, \mu_{l_1}, ..., \mu_{l_m})$ for which we have $d(\mu_{k_s}, \mu_{l_m}) < \beta$ and $|S \setminus T| \geq n$. Suppose now $t \in \{0, 1, 2, ..., s-n\}$ is such that $\mu_{k_t} \in T$ and $\mu_{k_{t+1}} \notin T$ (we set $\mu_{k_0} = \mu_0$). We define the function f_n on N via the formula

$$f_n(x) = \begin{cases} \frac{\alpha}{2} & x = \mu_{k_j} \text{ for } j \text{ odd}, j > t\,, \\ \frac{-\alpha}{2} & x = \mu_{k_j} \text{ for } j \text{ even}, j > t\,, \\ 0 & \text{else}\,. \end{cases}$$

Clearly, $f_n(\mu_0) = 0$ and $\|f_n\| \leq 1$. For the following choice of sings $\varepsilon_0 = 1$,

$$\varepsilon_i = \begin{cases} -\varepsilon_{i-1} & i = k_j \text{ for some } j \in \mathbb{N}\,, \\ \varepsilon_{i-1} & \text{else}\,, \end{cases}$$

we have

$$\sum_{i=0}^{k_s} \varepsilon_i (P_{i+1}^* - P_i^*) = -P_0^* + 2\sum_{j=1}^{s} (-1)^{j+1} P_{k_j}^* + (-1)^s P_{k_s+1}^* =: P$$

and then

$$\|P\| \geq \|Pf_n\| \geq \left\| \frac{Pf_n(\mu_{k_s}) - Pf_n(\mu_{l_m})}{d(\mu_{k_s}, \mu_{l_m})} \right\| \geq \frac{1}{\beta} \|Pf_n(\mu_{k_s}) - Pf_n(\mu_{l_m})\| =$$
$$= \frac{1}{\beta} \left| -f_n(0) + 2\sum_{j=1}^{s} (-1)^{j+1} f_n(\mu_{k_j}) + (-1)^s f_n(\mu_{k_s}) + 0 \right|$$
$$= \frac{1}{\beta} \left| 2 \sum_{j=t+1}^{s} \frac{\alpha}{2} - \frac{\alpha}{2} \right| \geq \frac{\alpha(s-t-1)}{\beta} \geq \frac{\alpha(n-1)}{\beta}.$$

□

Recall that a subset S of a metric space M is called an α, β-net whenever S is α-separated and β-dense in M, i.e. $\inf_{x \neq y} d(x, y) \geq \alpha$, $x, y \in S$ and $\sup_{x \in M} d(x, S) \leq \beta$.

In [4], the authors constructed a system of retractions on the integer lattice in c_0 which satisfies the conditions of Corollary 2. Through suitable homomorphisms they further showed the existence of a basis on any Free space of a net in a separable $C(K)$ space or a net in c_0^+, the positive cone in c_0.

Corollary 9. *Let N be a net in any of the following metric spaces: $C(K)$, K metrizable compact, or c_0^+ (the subset of c_0 consisting of elements with non-negative coordinates). The basis on $\mathcal{F}(N)$ constructed in* [4] *is conditional.*

Proof. First we consider the case $N = \mathbb{Z}^{<\omega} \subseteq c_0$, the integer lattice in c_0. Following the proof of Lemma 14 in [4] we see, there are chains which go parallelly along the first coordinate axis (or any other coordinate axis). Every such two chains hence satisfy the conditions of the previous lemma, which yields that a basis arising from these retractions cannot be unconditional. As the existence of bases in other cases than N being the integer lattice in c_0 was proven only by isomorphisms, we conclude that none of them are unconditional. □

Theorem 10. *Let N be an α, β-net in a finite-dimensional normed space X with $\dim X \geq 2$. Let $E = \{e_i\}_{i=1}^{\infty}$ be a retractional Schauder basis on $\mathcal{F}(N)$. Then E is conditional.*

In the following, $B_\varepsilon(x)$ denotes closed ball of radius $\varepsilon > 0$ and centre $x \in X$, $B_\varepsilon^\circ(x)$ denotes its interior. In the same way $B_\varepsilon := B_\varepsilon(0)$ and S_ε denotes sphere of radius ε and centre 0.

Proof. Let $\varphi_i : N \to N$ be the corresponding retractions to the basis E. We prove the theorem by showing that the assumptions of Lemma 8 are met. Denote $\sup_{i \in \mathbb{N}} \operatorname{Lip} \varphi_i = K < \infty$. Pick $n \in \mathbb{N}$, such that $n > 8K$. Define annulus with radii r and w, $w < r$ as $A(r, w) = B_{r+w}(0) \setminus B_{r-w}^\circ(0)$. Our aim is to prove there exist chains T, Z with final points $t, z \in A(3K\beta n + \beta, \beta) \cap N$ with $d(t, z) \leq 2\beta$ such that $x \in B_{K\beta n}$ holds for $x = x_{t,z}$, the final point of the chain $T \cap Z$. Then we have $d(t, x) \geq 3K\beta n - K\beta n = 2K\beta n$ and Step lemma 4 yields $|T \setminus Z| \geq n$, which by Lemma 8 concludes the proof.

For the following, for every two points $x, y \in N$ with $x \prec y$ denote T_x^y the chain with initial point x and final point y. Assume now for contradiction, for every pair of points $t, z \in A(3K\beta n + \beta, \beta) \cap N =: A$ with $d(t, z) \leq 2\beta$ the final point $x_{t,z}$ of $T_0^t \cap T_0^z$ lies outside the ball $B_{K\beta n}$. That means there exists a point $\mu_m \in N$ for some $m \in \mathbb{N}$ with $d(0, \mu_m) > K\beta n$ such that $\varphi_m(t) = \mu_m$ for every $t \in A$. To prove this, note that $0 \in \bigcap_{t \in A} T_0^t$ and as $\bigcap_{t \in A} T_0^t$ is a chain, it has a final point which we denote μ_m and prove that $d(0, \mu_m) > K\beta n$. We show that for every two points $t, z \in A$ the final point $x_{t,z}$ of the chain $T_0^t \cap T_0^z$ is of greater norm than $K\beta n$. Clearly, if $d(t, z) \leq 2\beta$, the statement holds as assumed. If $d(t, z) > 2\beta$, we can find a finite sequence of points $y_1, ..., y_l \in A$, $l \in \mathbb{N}$ such that $d(y_i, y_{i+1}) \leq 2\beta$ for every $i \in \{1, ..., l-1\}$ and that $y_1 = t$ and $y_l = z$. Then $x_{t,z} \in \left\{x_{y_i, y_{i+1}} |\ i \in \{1, ..., l-1\}\right\}$, which means $\|x_{t,z}\| > K\beta n$. Note that for any three points $s, t, z \in A$ the final point $x_{s,t,z}$ of the chain $T_0^s \cap T_0^t \cap T_0^z$ is equal to one of the points $x_{s,t}, x_{t,z}, x_{s,z}$. Indeed, as $x_{s,t}, x_{t,z} \in T_0^t$, we have that either $x_{s,t} \prec x_{t,z}$ or $x_{s,t} \succ x_{t,z}$. If $x_{s,t} \prec x_{t,z}$, then $x_{s,z} = x_{s,t} = x_{s,t,z}$ and the other case follows symmetrically. But from that we get inductively that for any finite number of points $t_1, ..., t_v$, there are indices $i, j \in \{1, ..., v\}$, such that the final point $x_{t_1,...,t_v}$ of the chain $\bigcap_{l=1} T_0^{t_l}$ equals x_{t_i,t_j}. Because for each two $t, z \in A$ we have $\|x_{t,z}\| > K\beta n$ and A is finite, we have $\|\mu_m\| > K\beta n$.

Observe further, that $T_{\mu_m}^t \cap B_{\beta n} = \emptyset$ holds for every chain $T_{\mu_m}^t$ with initial point μ_m and final point $t \in A$. Indeed, if $\mu_p \in T_{\mu_m}^t$, $p \in \mathbb{N}$ is such that $\|\mu_p\| \leq \beta n$, we have $\operatorname{Lip} \varphi_m \geq \frac{\|\varphi_m(0) - \varphi_m(\mu_p)\|}{\|\mu_p\|} = \frac{\|\mu_m\|}{\|\mu_p\|} > \frac{K\beta n}{\beta n} = K$, which is not possible.

Let us denote $S = \bigcup_{t \in A} T_{\mu_m}^t$ the set of all chains from μ_m to points of A. Let $S = \{\mu_{k_1}, ..., \mu_{k_q}\}$ for some $k_1 < k_2 < ... < k_q$, $q \in \mathbb{N}$. Note that $\mu_{k_1} = \mu_m$. For every chain $T = (t_1, ..., t_l)$, $l \in \mathbb{N}$ define a trajectory of the chain $\operatorname{Tr}(T)$ as the union of the line segments $\bigcup_{i=1}^{l-1} [t_i, t_{i+1}]$. Denote $D = \bigcup_{t \in A} \operatorname{Tr}\left(T_{\mu_m}^t\right)$. Define now a function $F : [1, q] \times A \to D$ via

$$F(t, x) = (t - i)\varphi_{k_{i+1}}(x) + (1 - (t - i))\varphi_{k_i}(x), x \in A, t \in [i, i + 1), i \in \{1, ..., q - 1\}$$

and $F(q, x) = x$, $x \in A$. We see that for each $t \in [1, q]$, the function $F(t, \cdot)$ is K-Lipschitz and that we have $F(1, x) = \mu_m$ for all $x \in A$.

Let $\varepsilon = 3K\beta n + \beta$ and consider $B = \left\{B_{2\beta}^o(x) \cap S_\varepsilon\right\}_{x \in A}$ as an open cover of S_ε and find a partition of unity $\{\psi_a\}_{a \in A}$ subordinated to the cover B. Define a function $R : [1, q] \times S_\varepsilon \to X$ by

$$R(t, x) = \sum_{a \in A} \psi_a(x) F(t, a), \quad t \in [1, q], \ x \in S_\varepsilon.$$

We see, that $R(1, x) = \mu_m$ for all $x \in S_\varepsilon$ and that

$$\sup_{x \in S_\varepsilon} |R(q, x) - x| \leq 2\beta.$$

Of course, R is continuous on $[1, q] \times S_\varepsilon$. Our goal is to prove there is a continuous deformation of S_ε into one point μ_m avoiding the origin, which is a contradiction. For that we define a straight-line homotopy between identity and $R(q, \cdot)$ by $W : [0, 1] \times S_\varepsilon \to A(\varepsilon, 2\beta)$, $W(t, x) = tR(q, x) + (1 - t)x$. Joining mappings W and R we get a mapping $Z : [0, q] \times S_\varepsilon \to X$ precisely defined by

$$Z(t, x) = \begin{cases} W(t, x) & t \in [0, 1), x \in S_\varepsilon \,, \\ R\left(\frac{q}{t}, x\right) & t \in [1, q], x \in S_\varepsilon \,. \end{cases}$$

All there is left to prove is that $R([1, q] \times S_\varepsilon) \cap \{0\} = \emptyset$. To see that, note that the value $R(t, x)$ is a convex combination of values $F(t, a)$, where $a \in A$ are such that $d(x, a) < 2\beta$. Fix therefore $x \in S_\varepsilon$ and let $\mu_{l_1}, \mu_{l_2}, ..., \mu_{l_p} \in A$ be such that $d(x, \mu_{l_i}) < 2\beta$ for all i. From the preceding paragraphs it follows that the trajectory $\operatorname{Tr}(T_{\mu_m}^{\mu_{l_i}})$ of each chain from μ_m to μ_{l_i} has no intersection with $B_{\beta \frac{n}{2}}$. Indeed, as the chain $T_{\mu_m}^{\mu_{l_i}}$ avoids the ball $B_{\beta n}$ and the distance between two consecutive points in a chain is bounded by $2\beta K$ and $n > 2K$, we get the result. From the fact that $d(\mu_{l_j}, \mu_{l_i}) \leq 4\beta$ for all i, j, we have that $\|F(t, \mu_{l_i}) - F(t, \mu_{l_j})\| \leq 4K\beta$

for all $t \in [1, q]$. But as $n > 8K$ we get $R(t, x) \neq 0$ for any $t \in [0, 1]$. Altogether we obtain $Z([0, q] \times S_\varepsilon) \cap \{0\} = \emptyset$, which was to prove.

□

One could ask in general what are the metric spaces M such that $\mathcal{F}(M)$ has an unconditional Schauder basis. It is clear that if M contains a line segment, then L_1 is contained in $\mathcal{F}(M)$ and therefore $\mathcal{F}(M)$ cannot have an unconditional Schauder basis. The only interesting cases are then topologically discrete spaces M. Our guess is that if $\mathcal{F}(M)$ has an unconditional Schauder basis, it is isomorphic to ℓ_1.

Open problem 1 *Suppose $\mathcal{F}(M)$ has an unconditional Schauder basis. Is it isomorphic to ℓ_1?*

In [15], one sees that $\mathcal{F}(M)$ is a complemented subspace of L_1 if and only if M can be bi-Lipschitzly embedded into an $\mathbb{R}$-tree. A complemented subspace of L_1 with unconditional basis is isomorphic to the space ℓ_1 due to [16]. One can therefore restate the conjecture above into: Suppose $\mathcal{F}(M)$ has a Schauder basis B. If M cannot be embedded into an $\mathbb{R}$-tree, is it true that B is conditional?

Open problem 2 *Is it true that for every uniformly discrete set $N \subseteq \mathbb{R}^2$ the space $\mathcal{F}(N)$ has a Schauder basis?*

It follows from Corollary 6 the answer is no if we restrict ourselves only to retractional Schauder bases. However, we don't know if, supposed the answer is yes, we can find for every uniformly discrete set $N \subseteq \mathbb{R}^2$ an extensional Schauder basis on $\mathcal{F}(N)$.

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